Recitation 05: Differentiation

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Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

Topics to be covered: (Nov 09-13)

2.8 The Mean-Value Theorem
2.9 Implicit Differentiation
Ch 3: Transcendental Functions
3.1 Inverse Functions
3.2 Exponential and Logarithmic Functions
3.3 The Natural Logarithm and Exponential

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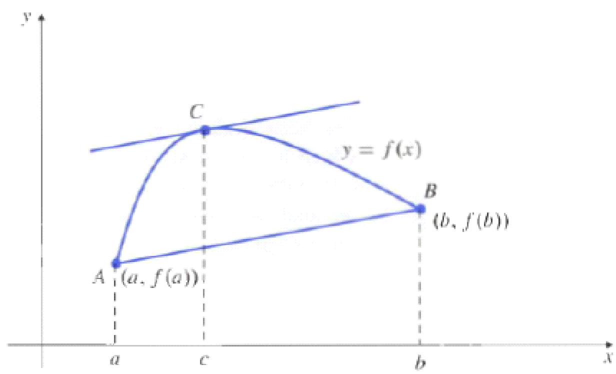
THEOREM: (The Mean Value Theorem)

Suppose that the function f is continuous on the closed, finite interval $[a, b]$ and that it is differentiable on the open interval (a, b). Then there exists a point c in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

This says that the slope of the chord line joining the points $(a, f(a))$ and $(b, f(b))$ is equal to the slope of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$, so the two lines are parallel.



DEFINITION: (Increasing and decreasing functions)

Suppose that the function f is defined on an interval I and that x_1 and x_2 are two points of I.

(a) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, we say f is **increasing** on I.
(b) If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, we say f is **decreasing** on I.
(c) If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, we say f is **nondecreasing** on I.
(d) If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, we say f is **nonincreasing** on I.

THEOREM:

Let J be an open interval, and let I be an interval consisting of all the points in J and possibly one or both of the endpoints of J. Suppose that f is continuous on I and differentiable on J.

(a) If $f'(x) > 0$ for all x in J, then f is increasing on I.

(b) If $f'(x) < 0$ for all x in J, then f is decreasing on I.

(c) If $f'(x) \geq 0$ for all x in J, then f is nondecreasing on I.

(d) If $f'(x) \leq 0$ for all x in J, then f is nonincreasing on I.

THEOREM:

If f is defined on an open interval (a, b) and achieves a maximum (or minimum) value at the point c in (a, b), and if $f'(c)$ exists, then $f'(c) = 0$. (Values of x where $f'(x) = 0$ are called **critical points** of the function f.)

THEOREM: (Rolle's Thm)

Suppose that the function g is continuous on the closed, finite interval $[a, b]$ and that it is differentiable on the open interval (a, b). If $g(a) = g(b)$, then there exists a point c in the open interval (a, b) such that $g'(c) = 0$.

DEFINITION: (one-to-one)

A function f is **one-to-one** if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$, or, equivalently, if

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2.$$

DEFINITION: (inverse func.)

If f is one-to-one, then it has an **inverse function** f^{-1}. The value of $f^{-1}(x)$ is the unique number y in the domain of f for which $f(y) = x$. Thus,

$$y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y).$$

SOME PROPERTIES

Properties of inverse functions

1. $y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y)$.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.
4. $f^{-1}(f(x)) = x$ for all x in the domain of f.
5. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
6. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f.
7. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$.

Laws of exponents

If $a > 0$ and $b > 0$, and x and y are any real numbers, then

(i) $a^0 = 1$ (ii) $a^{x+y} = a^x a^y$

(iii) $a^{-x} = \dfrac{1}{a^x}$ (iv) $a^{x-y} = \dfrac{a^x}{a^y}$

(v) $(a^x)^y = a^{xy}$ (vi) $(ab)^x = a^x b^x$

Laws of logarithms

If $x > 0$, $y > 0$, $a > 0$, $b > 0$, $a \neq 1$, and $b \neq 1$, then

(i) $\log_a 1 = 0$

(ii) $\log_a(xy) = \log_a x + \log_a y$

(iii) $\log_a \left(\frac{1}{x}\right) = -\log_a x$

(iv) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

(v) $\log_a \left(x^y\right) = y \log_a x$

(vi) $\log_a x = \dfrac{\log_b x}{\log_b a}$

If $a > 1$, then $\lim_{x \to 0+} \log_a x = -\infty$ and $\lim_{x \to \infty} \log_a x = \infty$.

If $0 < a < 1$, then $\lim_{x \to 0+} \log_a x = \infty$ and $\lim_{x \to \infty} \log_a x = -\infty$.

(i) $\ln(xy) = \ln x + \ln y$

(ii) $\ln \left(\frac{1}{x}\right) = -\ln x$

(iii) $\ln \left(\frac{x}{y}\right) = \ln x - \ln y$

(iv) $\ln \left(x^r\right) = r \ln x$

(i) $(\exp x)^r = \exp(rx)$

(ii) $\exp(x+y) = (\exp x)(\exp y)$

(iii) $\exp(-x) = \dfrac{1}{\exp(x)}$

(iv) $\exp(x - y) = \dfrac{\exp x}{\exp y}$

For the moment, identity (i) is asserted only for rational numbers r.

Question 1

12 Kasım 2020 Perşembe 11:31

Show that $x^3 + x^2 + 3x + 7 = 0$ has $\boxed{\text{exactly}}$ one real root.

IVT: f is $\boxed{\text{cont}}$ on $[a,b]$ and d is between $f(a)$ and $f(b)$

then: $\exists c \in (a,b) \ni$

$f(c) = d$.

MVT: f is $\boxed{\text{cont}}$ on $[a,b]$

f is $\boxed{\text{diff}}$ on (a,b)

then: $\exists c \in (a,b) \ni$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Solution:

Define $f(x) = x^3 + x^2 + 3x + 7$.

- f is cont on $\mathbb{R}$
- f is diff on $\mathbb{R}$

> Since it is a polyn func.

First, let's show that the func has at least one root:

$f(0) = 7 > 0$

$f(-3) = -20 < 0$

> f is cont on $[-3,0] \subset \mathbb{R}$ and $f(0) > 0$, $f(-3) < 0$ by IVT;

we have $\exists x_0 \in (-3, 0) \ni f(x_0) = 0$

$\Rightarrow f$ has $\boxed{\text{at least}}$ one root.

Assumption: f has two roots say x_0 and x_1 s.t $x_0 \neq x_1$.

$f(x_0) = 0$ and $f(x_1) = 0$ s.t $x_0 \neq x_1$. (WLOG $x_0 < x_1$) $x_1 < x_0$

without loss of generality

- f is cont on $[x_0, x_1] \subset \mathbb{R}$
- f is diff. on $(x_0, x_1) \subset \mathbb{R}$

> By MVT:

$$\exists c \in (x_0, x_1) \ni f'(c) = \frac{\overset{0}{f(x_1)} - \overset{0}{f(x_0)}}{\underbrace{x_1 - x_0}_{\neq 0}} = 0$$

$$\boxed{\Rightarrow \exists c \in (x_0, x_1) \ni f'(c) = 0}$$ → result of the assumption

On the other hand, $f(x) = x^3 + x^2 + 3x + 7$

$\Rightarrow f'(x) = 3x^2 + 2x + 3$

Recall: $ax^2 + bx + c = 0$

$-b \mp \sqrt{\Delta}$

reality. ✓

$\Rightarrow f'(x) = 3x^2 + 2x + 3$

$\Delta = b^2 - 4ac = 4 - 4.3.3 = -32 < 0$

f' has not any real roots.

Recall: $ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$$

if $\Delta < 0 \Rightarrow x_{1,2} \notin \mathbb{R}$

We have a contradiction ↯

$\Rightarrow$ Assumption is wrong.

$\Rightarrow f$ cannot have two real roots.

$\Rightarrow f$ has exactly one real roots. (Since we ~~al~~ already prove f has at least one root.

Question 2

12 Kasım 2020 Perşembe 11:31

Prove that

$e^x \geq x+1$ for all x

$x=0$
$x>0$
$x<0$

MVT:

f is cont on $[a,b]$
f is diff. on (a,b) then

$\exists c \in (a,b)$ $\ni$

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Solution:

case 1: $x=0$

$e^0 = 1 \geq 0+1$ ✓

case 2: $x>0$

Define $f(x) = e^x - x - 1$. f is cont and diff on $\mathbb{R}$

since it is an addition of exp. and poly func!

f is cont on $[0,x] \subset \mathbb{R}$
f is diff on $(0,x) \subset \mathbb{R}$ By MVT

$\exists c \in (0,x)$ $\ni$ $f'(c) = \dfrac{f(x) - \overset{0}{f(0)}}{x-0} = \dfrac{e^x - x - 1}{x}$

We have $f'(x) = e^x - 1 \Rightarrow f'(c) = e^c - 1$,

$c > 0. \Rightarrow e^c > e^0 = 1$ (exp is incres)

$\Rightarrow e^c - 1 > 0 \Rightarrow f'(c) > 0$,

By MVT $f'(c) = \dfrac{e^x - x - 1}{x} > 0 \Rightarrow e^x - x - 1 > 0$ ✓

$x>0$

$\Rightarrow \boxed{e^x > x+1}$

case 3: $x<0$

Similarly, f is cont on $[x,0] \subset \mathbb{R}$
f is diff on $(x,0) \subset \mathbb{R}$

By MVT. $\exists c \in (x,0)$ $\ni$

By MVT, $\exists c \in (x,0)$ $\ni$

$$f'(c) = \frac{f(0)-f(x)}{0-x} = \boxed{\frac{-e^x+x+1}{-x}} < 0$$

$f'(c) = e^c - 1$ and $\underbrace{c<0}$ $\Rightarrow e^c < e^0 = 1$ (exp is inc.)

$\Rightarrow e^c - 1 < 0$

$\Rightarrow \underbrace{f'(c) < 0}$

$x<0$
$-x>0$

$$\frac{-e^x+x+1}{-x} < 0 \Rightarrow -e^x + x + 1 < 0$$

$\Rightarrow \boxed{e^x > x+1}$ ✓

Finally: if
$x=0 \Rightarrow e^x = x+1$
$x<0 \Rightarrow e^x > x+1$
$x>0 \Rightarrow e^x > x+1$

$\Rightarrow$ $\boxed{e^x \geq x+1 \text{ for all } x.}$

Question 3

12 Kasım 2020 Perşembe 11:31

$c_1 \neq c_2$

Let f be a function defined on $[0,9]$ and f', f'' exist on $[0,9]$. If $f(1) = -2$, $f(3) = 5$, $f(4) = 6$ and $f(8) = 20$, then show that there is a number $c \in (0,9)$ such that $f''(c) = 0$.

(*) Recall:

f is diff. on $[a,b]$

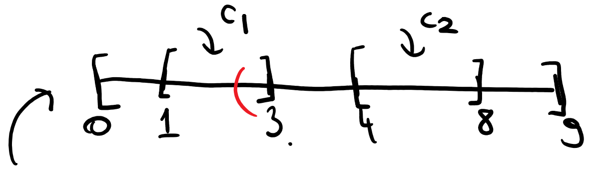
$\Downarrow$ ~~$\Uparrow$~~

f is cont on $[a,b]$

Solution:

$(f')'$

f' and f'' exist on $[0,9]$ $\Rightarrow$ f and f' are cont on $[0,9]$



$\Rightarrow$ First, f is cont on $[1,3] \subset [0,9]$

f is diff on $(1,3) \subset [0,9]$

By MVT $\exists c_1 \in (1,3) \ni$

$$f'(c_1) = \frac{\overset{5}{f(3)} - \overset{-2}{f(1)}}{3-1} = \frac{7}{2}$$

$\Rightarrow$ Second, f is cont on $[4,8] \subset [0,9]$

f is diff on $(4,8) \subset [0,9]$

By MVT $\exists c_2 \in (4,8) \ni$

$$f'(c_2) = \frac{\overset{20}{f(8)} - \overset{6}{f(4)}}{8-4} = \frac{7}{2}$$

$\Rightarrow$ Finally, let's apply MVT for f'.

f' is cont on $[c_1, c_2] \subset [0,9]$

f' is diff on $(c_1, c_2) \subset [0,9]$

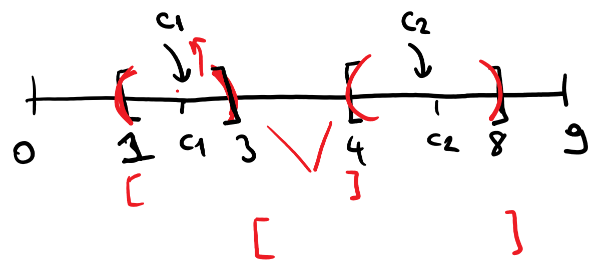
By MVT,

$\subset (0,9)$

$\exists c \in (c_1, c_2) \ni$ $7/2$ $7/2$

$\exists c \in (c_1, c_2) \ni$

$$f''(c) = \frac{\overbrace{f'(c_2)}^{7/2} - \overbrace{f'(c_1)}^{7/2}}{\underbrace{c_2 - c_1}_{\neq 0}} = 0$$



$\Rightarrow c_1 \neq c_2$ since $c_1 \in (1,3)$ and $c_2 \in (4,8)$

$(1,3) \cap (4,8) = \emptyset$ ✓

$$\boxed{\text{As a result, } \exists c \in (c_1, c_2) \subset (0,9) \ni f''(c) = 0}$$

Question 4

12 Kasım 2020 Perşembe 11:31

Given $f(x) = \underbrace{x^5 + 7x}\, - \overbrace{2\sin(\pi x)} - 2$,

(a) Show that $f^{-1}(x)$ exists.

(b) Find the domain and range of $f^{-1}(x)$.

(c) Compute $\frac{df^{-1}}{dx}(6)$

Recall :

① if f is 1-1 $\Rightarrow$ f^{-1} exist.

② if f is (str.) decr. or incr.
$\Rightarrow$ f is 1-1.

③ if $f' < 0 \Rightarrow f$ is decr.
$f' > 0 \Rightarrow f$ is incr.

Solution:

a) $f'(x) = \overbrace{5x^4}^{\geq 0} + \overbrace{7 - 2\pi\cos(\pi x)}^{>0} > 0$ for all x.

$f'(1) = 12 + 2\pi$

$-1 \leq \cos(\pi x) \leq 1$

$-2\pi \leq -2\pi \cdot \cos(\pi x) \leq 2\pi$

$0 < 7 - 2\overset{3.14\ldots}{\pi} \leq 7 - 2\pi\cos(\pi x) \leq 7 + 2\pi$

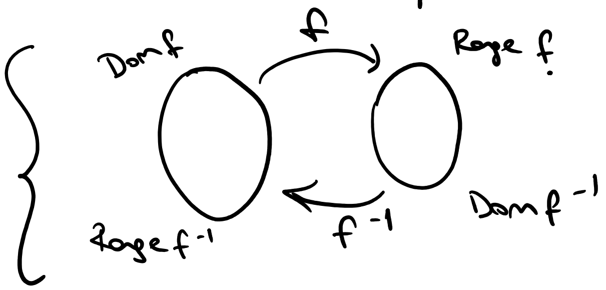
$\Rightarrow f'(x) > 0$ for all x.

$\Rightarrow f$ is strictly increasing

$\Rightarrow f$ is 1-1

$\Rightarrow f^{-1}$ exist.

b) Recall : $\boxed{\text{Dom } f = \text{Range } f^{-1}}$ $\boxed{\text{Range } f = \text{Dom } f^{-1}}$



$\text{Dom } f = \mathbb{R} \Rightarrow \text{Range } f^{-1} = \mathbb{R}$

$\text{Range } f = \mathbb{R} \Rightarrow \text{Dom } f^{-1} = \mathbb{R}$

c) $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=6} = ?$

defn.

$\dfrac{d}{dx} \Big|_{x=6} = \,?$

$$y = f^{-1}(x) \iff \boxed{x = f(y)}$$ defn.

We look for $\frac{dy}{dx}$: To calculate it :

; $x = f(y)$ Let's take derivative of both sides wrt. x.

$\Rightarrow 1 = f'(y) \cdot y'$ (chain rule)

$$\Rightarrow y' = \frac{1}{f'(y)} \qquad \boxed{y = f^{-1}(x)}$$

$$\boxed{\Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

formula for derivative of the inverse func.

$$\Rightarrow \frac{d}{dx} f^{-1}(6) = \boxed{\frac{1}{f'(\underbrace{\boxed{f^{-1}(6)}}_{c=1})}} = \frac{1}{f'(1)} = \frac{1}{12+2\pi}$$

answer.

$$f^{-1}(6) = c \iff 6 = f(c)$$

$$f(c) = \boxed{c^5 + 7c - 2\sin(\pi c) - 2} = 6$$

$$\Rightarrow \boxed{c = 1}$$ guess

Exercise 1

12 Kasım 2020 Perşembe 11:31

Let $f(x) = x^4 + x^2 + x - 10$.

i. Show that f has at least 2 real roots.

ii. Show that f does **not** have 3 real roots or more.

Exercise 2

12 Kasım 2020 Perşembe 11:31

Prove the following inequality:

$e^x \geq x + 1 + \frac{x^2}{2}$ for all $x \geq 0$.

Exercise 3

12 Kasım 2020 Perşembe 21:01

Prove the following inequality:

$\tan x > x$ for $0 < x < \pi/2.$

Exercise 4

12 Kasım 2020 Perşembe 21:02

Solve the following problem:

If $f(1) = 10$ and $f'(x) \geq 2$ for all x, then show that $f(4) \geq 16$.

Exercise 5

12 Kasım 2020 Perşembe 21:07

If $xy + y^3 = 1$, find the value of y'' at the point where $x = 0$.