Recitation 04: Differentiation

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Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

Topics to be covered: (Nov 02-06)

2.1 Tangent Lines and Their Slope

2.2 The Derivative

2.3 Differentiation Rules

2.4 The Chain Rule

2.5 Derivatives of Trigonometric Functions

2.6 Higher-Order Derivatives

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DEFINITION:

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

THEOREM:

Differentiability implies continuity

If f is differentiable at x, then f is continuous at x.

RULES:

$$(f+g)'(x) = f'(x) + g'(x),$$
$$(f-g)'(x) = f'(x) - g'(x),$$
$$(Cf)'(x) = Cf'(x).$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}.$$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

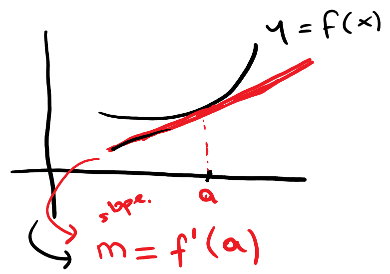
$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

Question 1

5 Kasım 2020 Perşembe 10:29

(a) Consider the curve given by $\boxed{y = \frac{16}{x} - x^2}$. Find the points where this curve has a horizontal tangent line. $16.x^{-1}$ a

ex: (b) For what values(s) of the constant k do the curves $y = kx^2$ and $y = k(x-2)^2$ intersect at right angles?



Solution :

a) Horizontal lines have the slope of 0. $(m=0)$

$$y' = \frac{-16}{x^2} - 2x \Rightarrow \boxed{\frac{-16}{a^2} - 2a} = 0$$

$$\Rightarrow \frac{-16-2a^3}{a^2} = 0 \Rightarrow \boxed{a=-2} \qquad (-2,-12)$$

b) If two curves intersect (at a point a) at a right angle $\Rightarrow$

the tangent lines of the curves (at the point a) are perpendicular.

$$\Rightarrow \boxed{m_1 \cdot m_2 = -1} \ (1)$$

at point a

$$\begin{cases} y = kx^2 & \Rightarrow y' = 2kx & \Rightarrow \boxed{m_1 = 2ka} \\ y = k.(x-2)^2 & \Rightarrow y' = 2k.(x-2) & \Rightarrow \boxed{m_2 = 2k.(a-2)} \end{cases}$$

Two curves intersect at the point a.

$$(2)\ \boxed{k.a^2 = k.(a-2)^2} \Rightarrow ka^2 - k.(a-2)^2 = 0$$

$$k.(a-a+2)(a+a-2) = 0$$

$$2k.(2a-2) = 0$$

$$\Rightarrow \cancel{\boxed{k=0}} \quad \text{OR} \quad \boxed{a=1}$$

$$\boxed{a=1}$$

$$(1)\ \boxed{m_1.m_2 = -1} \Rightarrow 2ka.\ 2k.(a-2) = -1$$

$$\Rightarrow 2k.2k.(-1) = -1$$

$$\Rightarrow 4k^2 = 1$$

$$\Rightarrow k = \mp\frac{1}{2}$$

Question 2

5 Kasım 2020 Perşembe 12:55

Calculate the derivative of the given function using the definition of the derivative.

(a) $F(x) = \frac{1}{\sqrt{1+x^2}}$

ex. (b) $f(x) = x^{1/3}$

① roud of limit ∞ $-\infty$ ② left ≠ right

if limit DNE f' is undefn. (DNE)

Recall : (Defn of derivative)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{if limit exist!!}$$

Solution :

a) $$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}}{h}$$

$$\boxed{a^2-b^2=(a-b)(a+b)}$$

$$= \lim_{h\to 0} \frac{\overbrace{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}}{h.\sqrt{1+x^2}.\sqrt{1+(x+h)^2}} \cdot \frac{\overbrace{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$= \lim_{h\to 0} \frac{\cancel{1}+x^2 - \cancel{1} - (x+h)^2}{h.\sqrt{1+x^2}\,\sqrt{1+(x+h)^2}} \cdot \frac{1}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\overbrace{(\cancel{x} - \cancel{x} - h)}^{-\cancel{h}}\,(x+x+h)}{\cancel{h}.\sqrt{1+x^2}\,.\,\sqrt{1+(x+h)^2}} \cdot \frac{1}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$= \frac{-1.\,2x}{\sqrt{1+x^2}\,.\sqrt{1+x^2}} \cdot \frac{1}{\underbrace{\sqrt{1+x^2} + \sqrt{1+x^2}}_{2\sqrt{1+x^2}}} = \frac{-x}{(1+x^2)^{3/2}}$$

{ ∞, $-\infty$ ↳ limit DNE; $\lim_{h\to 0^+} \neq \lim_{h\to 0^-}$ ↳ limit DNE }

$$\Rightarrow F'(x) = \frac{-x}{(1+x^2)^{3/2}}$$

b) $f(x) = x^{1/3}$. Let's find the derivative of the func. using the limit defn of derivative:

$$\lim \frac{f(x+h)-f(x)}{h} = \lim \frac{\boxed{(x+h)^{1/3}} - \boxed{x^{1/3}}}{} \cdot \frac{(x+h)^{2/3} + (x+h)^{1/3}.x^{1/3} + x^{2/3}}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\boxed{(x+h)^{1/3}} - \boxed{x^{1/3}}}{h} \cdot \frac{(x+h)^{\cdot} + (x+h)\cdot x^{\cdot} + x^{\cdot}}{(x+h)^{2/3} + (x+h)^{1/3}\cdot x^{1/3} + x^{2/3}}$$

$$= \lim_{\substack{h\to 0 \\ h\neq 0}} \frac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}} \cdot \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}\cdot x^{1/3} + x^{2/3}}$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$= \frac{1}{x^{2/3} + \underbrace{x^{1/3}\cdot x^{1/3}}_{x^{2/3}} + x^{2/3}} = \frac{1}{3\cdot x^{2/3}} = \frac{1}{3}\cdot x^{-2/3}$$

$$\Rightarrow \boxed{f'(x) = \frac{x^{-2/3}}{3}}$$

Question 3

5 Kasım 2020 Perşembe 12:57

How should the function $f(x) = x^2 \sin(\frac{1}{x})$ be defined at $x = 0$ so that it is continuous at $x = 0$? Is it then differentiable there?

Recall:

f is cont at x_0

$$\lim_{x \to x_0} f(x) = f(x_0)$$

exist and eq.

Solution:

$f(x) = x^2 . \sin(\frac{1}{x})$ does not defined at $x=0$. The func. is not cont at $x=0$. Let's check.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 . \sin(\frac{1}{x}) = 0$$

Since we have the following:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 . \sin\left(\frac{1}{x}\right) \leq x^2$$

and

$$\left.\begin{array}{l} \lim_{x \to 0} x^2 = 0 \\ \lim_{x \to 0} -x^2 = 0 \end{array}\right]$$

By squeeze thm

If we define the func as follows:

$$F(x) = \begin{cases} x^2 . \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is given in quest.

new func. we defined.

$$\lim_{x \to 0} F(x) = 0 = F(0) \Rightarrow$$

F func is cont at $x=0$.

Recall: f is diff $\Rightarrow$ f is cont.

(f is cont $\cancel{\Rightarrow}$ f is dif) $\Leftarrow$ not true.

$\Rightarrow$ new func.

Let's check the func is dif. or not at $x=0$

Using the limit defn of derivative:

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \cdot \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \cdot \sin\left(\frac{1}{h}\right) = 0$$ since;

$$-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$$
$$-h \leq h \cdot \sin\left(\frac{1}{h}\right) \leq h$$

$$\left.\begin{array}{l} \lim_{h \to 0} h = 0 \\ \lim_{h \to 0} -h = 0 \end{array}\right]$$ By squeeze thm.

$\Rightarrow f'(0) = 0$. The func is diff at $x = 0$.

Question 4

5 Kasım 2020 Perşembe 12:58

Let $g(x)$ be continuous at $x = a$ and consider the function $f(x) = \underbrace{(x-a)g(x)}_{f(a)=0}$. Find $f'(a)$ in terms of g.

Solution:

We can not use the product rule to calculate the derivative of the func, since we donot know about the diff. of the func. g. (We know just only g is cont at $x=a$)

Using the limit defn of derivative;

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$x-a=h$
$x\to a$
$h\to 0$

$$= \lim_{x\to a}\frac{(x-a)g(x)-0}{x-a} = \lim_{x\to a} g(x) = g(a)$$

Since g is cont $x=a$.

$$\boxed{f'(a) = g(a)}$$

Question 5

5 Kasım 2020 Perşembe 12:59

Given that f(1) = 2; f'(1) = 1; g(1) = 3; g'(1) = 4, calculate the following:

(a) $\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Big|_{x=1}$

(b) $\frac{d}{dx}(x^3 f(x))\Big|_{x=1}$

(c) $\frac{d}{dx}(f^2(x)g(x))\Big|_{x=1}$

Recall:

① $(f\cdot g)' = f'\cdot g + f\cdot g'$

② $\left(\frac{f}{g}\right)' = \frac{f'g - f\cdot g'}{g^2}$

chain rule

③ $[f(g(x))]' = f'(g(x))\cdot g'(x)$

Solution:

a) $$\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Bigg|_{x=1} = \frac{f'(x)[g(x)+x] - f(x)[g'(x)+1]}{[g(x)+x]^2}\Bigg|_{x=1}$$

$$= \frac{\overbrace{f'(1)}^{1}[\overbrace{g(1)}^{3}+1] - \overbrace{f(1)}^{2}[\overbrace{g'(1)}^{4}+1]}{[\underbrace{g(1)}_{3}+1]^2} = \frac{-3}{8}$$

(check it!)

b) $$\frac{d}{dx}(x^3.f(x))\Big|_{x=1} = 3x^2.f(x) + x^3.f'(x)\Big|_{x=1}$$

$$= 3.\underbrace{f(1)}_{2} + 1.\underbrace{f'(1)}_{1} = 7$$

c) $$\frac{d}{dx}\Big[\underbrace{f^2(x)}_{\parallel}.g(x)\Big]\Bigg|_{x=1} = \overbrace{2.f(x)\,f'(x)}\,.g(x) + f^2(x).g'(x)\Big|_{x=1}$$

$\hookrightarrow [f(x)]^2$

$$= 2.\underbrace{f(1)}_{2}.\underbrace{f'(1)}_{1}\,\underbrace{g(1)}_{3} + [\underbrace{f(1)}_{2}]^2\,\underbrace{g'(1)}_{4} = 28$$

Question 6

5 Kasım 2020 Perşembe 13:00

Find the derivative of the following functions:

(a) $f(x) = \sqrt{3x + \sqrt{2 + \sqrt{1-x}}}$

(b) $g(x) = \left(\frac{1+\sin 3x}{3-2x}\right)^{-1}$ $= -1$

ex: (c) $h(x) = \tan \frac{\pi}{\sqrt{25 - x^2}}$

Solution :

using chain rule.

a) $$f'(x) = \frac{1}{2} \cdot \left(3x + \sqrt{2 + \sqrt{1-x}}\right)^{-1/2} \cdot \left[3 + \frac{1}{2} \cdot \left(2 + \sqrt{1-x}\right)^{-1/2} \cdot \frac{1}{2} \cdot (1-x)^{-1/2} \cdot (-1)\right]$$

chain rule

b) $$g'(x) = -\left(\frac{1+\sin 3x}{3-2x}\right)^{-2} \cdot \left(\frac{1+\sin 3x}{3-2x}\right)'$$

$$= -\left(\frac{1+\sin 3x}{3-2x}\right)^{-2} \left(\frac{3\cos 3x(3-2x) - (-2).(1+\sin 3x)}{(3-2x)^2}\right)$$

Question 7

5 Kasım 2020 Perşembe 13:01

tangent line eqn $x=8$

a) Suppose f is a differentiable function and $y = x/4 - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point $x = 8$. If $g(x) = (f(x^3))^2$, find an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 2$.
b) If $g''(2) = 0$, find $f''(8)$.

Solution:

a) The tangent line to $y=f(x)$ at $x=8$ is given by

$$\boxed{y-y_0=m.(x-x_0)}$$

$$y=\frac{x}{4}-3$$

$$y-f(8)=f'(8).(x-8)$$

$x_0=8$

$y_0=f(x_0)=f(8)$

$m=f'(x_0)=f'(8)$

$$\frac{x}{4}-3=f'(8).x\overbrace{-8f'(8)+f(8)}$$

$$\boxed{f'(8)=\frac{1}{4}} \quad \text{and} \quad -3=-8\cdot\frac{1}{4}+f(8) \Rightarrow \boxed{f(8)=-1}$$

Let's find the tangent line to $y=g(x)$ at $x=2$.

$$y-y_0=m.(x-x_0)$$

$$\boxed{y-g(2)=g'(2).(x-2)}$$

$\Rightarrow$ we need to calculate $g(2)$ and $g'(2)$

$$g(x)=[f(x^3)]^2 \Rightarrow g'(x)=2.f(x^3).f'(x^3).3x^2$$

$$g(2)=[f(8)]^2=1 \quad \text{and} \quad g'(2)=2.\overset{-1}{f(8)}.\overset{1/4}{f'(8)}.3.4$$

$$g'(2)=-6$$

$$\Rightarrow \boxed{y-1=-6(x-2)}$$

b) $g(x)=[f(x^3)]^2$

$$\boxed{(f.g.h)'=f'g.h+f.g'.h+f.g.h'}$$

$$g'(x)=6.f(x^3).f'(x^3).\quad x^2$$

$$g''(\underset{2}{x}) = 6.\Big[f'(\underset{2}{x}^3).3\underset{2}{x}^2.f'(\underset{2}{x}^3).x^2 + f(x^3).f''(x^3).3x^2.x^2 + f(x^3).f'(x^3).2x\Big]\Big|_{x=2}$$

from the ques. $\boxed{g''(2)=0}$

$$0 = 6.\Big[\overbrace{f'(8)}^{1/4}.3.4.\overbrace{f'(8)}^{1/4}.4 + \overbrace{f(8)}^{-1}.f''(8).3.4.4 + \overbrace{f(8)}^{-1}.\overbrace{f'(8)}^{1/4}.2.2\Big]$$

$\Rightarrow f''(8) \Rightarrow$ you can calculate ☺