Recitation 04: Differentiation

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Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

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Topics to be covered: (Nov 02-06)

2.1 Tangent Lines and Their Slope
2.2 The Derivative
2.3 Differentiation Rules
2.4 The Chain Rule
2.5 Derivatives of Trigonometric Functions
2.6 Higher-Order Derivatives

DEFINITION:

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

THEOREM:

Differentiability implies continuity

If f is differentiable at x, then f is continuous at x.

RULES:

$$(f+g)'(x) = f'(x) + g'(x),$$
$$(f-g)'(x) = f'(x) - g'(x),$$
$$(Cf)'(x) = Cf'(x).$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

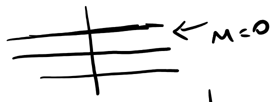
$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}.$$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

Question 1

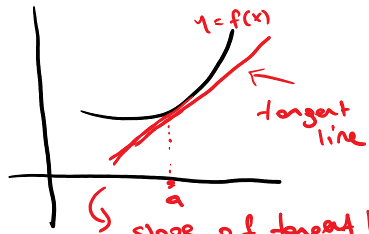
5 Kasım 2020 Perşembe 10:29



$16 \cdot x^{-1} - x^2$

(a) Consider the curve given by $y = \frac{16}{x} - x^2$. Find the points where this curve has a horizontal tangent line.

ex: (b) For what values(s) of the constant k do the curves $y = kx^2$ and $y = k(x-2)^2$ intersect at right angles?



slope of tangent line

$m = f'(a)$

Solution :

a) Horizontal tangent line has the slope of 0 !

$y' = \frac{-16}{x^2} - 2x$. Let the point a where this curve has a horizontal tangent line.

$-\frac{16}{a^2} - 2a = 0 \Rightarrow -16 - 2a^3 = 0 \Rightarrow a = -2 \qquad (-2, -12)$

b) If two curves intersect with right angle at the point a. $\Rightarrow$ the tangent lines of this curves at the point a are perpendicular $\Rightarrow$

$\boxed{m_1 \cdot m_2 = -1}$ (2)

$y = kx^2$ and $y = k \cdot (x-2)^2$ are the given curves.

$y' = 2kx \qquad y' = 2k \cdot (x-2)$

(1) $\boxed{ka^2 = k \cdot (a-2)^2}$ Since this curves intersect at $x = a$.

$ka^2 - k \cdot (a-2)^2 = 0 \Rightarrow k \cdot (\cancel{a} - \cancel{a} + 2)(a + a - 2) = 0$

$\Rightarrow \boxed{a = 1}$ or $\cancel{\boxed{k=0}}$

from eqn (2)

$y = kx^2 \Rightarrow y' = 2kx \Rightarrow m_1 = 2ka$

$y = k \cdot (x-2)^2 \Rightarrow y' = 2k \cdot (x-2) \Rightarrow m_1 = 2k \cdot (a-2)$

$\boxed{a = 1}$

$m_1 \cdot m_2 = -1 \Rightarrow \boxed{2ka \cdot 2k \cdot (a-2) = -1}$

$\Rightarrow 2k \cdot 1 \cdot 2k \cdot -1 = -1$

$\Rightarrow 4k^2 = 1 \Rightarrow \boxed{k = \mp \frac{1}{2}}$

Question 2

5 Kasım 2020 Perşembe 12:55

Calculate the derivative of the given function using the defition of the derivative.

(a) $F(x) = \frac{1}{\sqrt{1+x^2}}$

ex: (b) $f(x) = x^{1/3}$

if limit exist

$\Updownarrow$

- right = left
- not infinity

Recall: (Defn of derivative)

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Solution:

a)

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{h \cdot \sqrt{1+x^2} \cdot \sqrt{1+(x+h)^2}} \cdot \frac{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$\boxed{a^2 - b^2 = (a-b)(a+b)}$$

$$= \lim_{h \to 0} \frac{\underbrace{\cancel{1}+x^2 - (\cancel{1} + (x+h)^2)}_{= x^2 - (x+h)^2}}{h \cdot \sqrt{1+x^2} \cdot \sqrt{1+(x+h)^2}} \cdot \frac{1}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$= \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{(\underbrace{\cancel{x} - \cancel{x} - \cancel{h}}_{-1})(x + x + h)}{\cancel{h} \cdot \sqrt{1+x^2} \cdot \sqrt{1+(x+h)^2}} \cdot \frac{1}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$= \frac{-1 \cdot (2x)}{\sqrt{1+x^2} \cdot \sqrt{1+x^2}} \cdot \frac{1}{\underbrace{\sqrt{1+x^2} + \sqrt{1+x^2}}_{2\sqrt{1+x^2}}} = \frac{-x}{(1+x^2)^{3/2}} \quad //$$

$$F(x) = \frac{1}{\sqrt{1+x^2}} \Rightarrow F'(x) = \frac{-x}{(1+x^2)^{3/2}}$$

Question 3

5 Kasım 2020 Perşembe 12:57

How should the function $f(x) = x^2 \sin(\frac{1}{x})$ be defined at $x = 0$ so that it is continuous at $x = 0$? Is it then differentiable there?

Solution :

First, f is undefined at x=0. We want to make this func cont. therefore; the following must satisfy.

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0 \quad \text{Since we have}$$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \qquad \text{and} \qquad \left.\begin{array}{l} \lim_{x \to 0} x^2 = 0 \\ \lim_{x \to 0} -x^2 = 0 \end{array}\right] \text{(By squeeze thm.)}$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

Define,

$$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

x^{-1}

when $x \neq 0$

$$f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$$

product and chain

$$(f \cdot g)' = f' g + f \cdot g'$$

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

f is cont !

$x=0$ Let's check the func is diff. or not at x=0.

using the limit defn of derivative.

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\boxed{\begin{array}{l} x - x_0 = h \\ x \to x_0 \\ h \to 0 \end{array}}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{x^2 \cdot \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$$

$= 0$? Since,

$-1 \le \sin(\frac{1}{x}) \le 1$ $\qquad \lim_{x\to 0} x = \lim_{x\to 0} -x = 0$ (By squeez.thm)

$-x \le x\sin(\frac{1}{x}) \le x$

What if:

$\lim_{x\to 0} = \infty$ $= -\infty$

$f'(0)$ dNE

f is not diff. at $x=0$

$\boxed{f'(0) = 0}$. func is diff.

Question 4

5 Kasım 2020 Perşembe 12:58

(*) func is diff. on D.
$\Downarrow$
func is cont. on D.

(*) func is cont
$\cancel{\Downarrow}$
f is diff

Let $g(x)$ be continuous at $x = a$ and consider the function $\boxed{f(x) = (x-a)g(x)}$. Find $f'(a)$ in terms of g.

does not mean that

$f(a) = 0$

Solution:

(*) We can not use the product to calculate the derivative of func f at $x=a$ Since we donot know anything about diff of g. (we know just g is cont at $x=a$)

(*) Using limit defn of derivative.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\substack{x \to a \\ x \neq 0}} \frac{\cancel{(x-a)}g(x) - 0}{\cancel{x-a}}$$

$$= \lim_{x \to a} g(x) = g(a)$$ Since g is cont at $x=a$

$$\boxed{f'(a) = g(a)}$$

Question 5

5 Kasım 2020 Perşembe 12:59

Given that f(1) = 2; f'(1) = 1; g(1) = 3; g'(1) = 4, calculate the following:

(a) $\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Big|_{x=1}$

(b) $\frac{d}{dx}(x^3 f(x))\Big|_{x=1}$

(c) $\frac{d}{dx}(f^2(x)g(x))\Big|_{x=1}$

Recall:

① $(f\cdot g)' = f'\cdot g + f\cdot g'$

② $\left(\frac{f}{g}\right)' = \frac{f'g - f\cdot g'}{(g)^2}$

③ $[f(g)]' = f'(g(x))\cdot g'(x)$

Solution:

a) $\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Bigg|_{x=1} \overset{\text{from quo. rule}}{=} \frac{f'(x)\cdot[g(x)+x] - f(x)\cdot[g'(x)+1]}{[g(x)+x]^2}\Bigg|_{x=1}$

$= \frac{\overset{1}{f'(1)}\cdot[\overset{3}{g(1)}+1] - \overset{2}{f(1)}\,[\overset{4}{g'(1)}+1]}{[\underbrace{g(1)}_{3}+1]^2} = -\frac{3}{8}$ (check calc)

b) $\frac{d}{dx}\left(x^3\cdot f(x)\right)\Bigg|_{x=1} \overset{\text{product rule.}}{=} 3x^2\cdot f(x) + x^3\cdot f'(x)\Big|_{x=1}$

$= 3\cdot\underbrace{f(1)}_{2} + 1\cdot\underbrace{f'(1)}_{1} = 7$

c) $\frac{d}{dx}\left(\underbrace{f^2(x)}_{=[f(x)]^2}\,g(x)\right)\Bigg|_{x=1} \overset{\text{product and chain rules}}{=} 2\cdot f(x)\cdot f'(x)\cdot g(x) + [f(x)]^2\cdot g'(x)\Big|_{x=1}$

$= 2\cdot\underbrace{f(1)}_{2}\cdot\underbrace{f'(1)}_{1}\cdot\underbrace{g(1)}_{3} + (\underbrace{f(1)}_{2})^2\cdot\underbrace{g'(1)}_{4} = 28$

Question 6

5 Kasım 2020 Perşembe 13:00

Find the derivative of the following functions:

(a) $f(x) = \sqrt{3x + \sqrt{2 + \sqrt{1 - x}}}$

(b) $g(x) = \left(\dfrac{1 + \sin 3x}{3 - 2x}\right)^{-1}$

ex. (c) $h(x) = \tan \dfrac{\pi}{\sqrt{25 - x^2}}$

$$\left(\sqrt{f(x)}\right)' = \frac{1}{2}\left[f(x)\right]^{-1/2} \cdot f'(x)$$

Solution :

a) Using the chain rule we get:

$$f'(x) = \frac{1}{2} \cdot \left(3x + \sqrt{2 + \sqrt{1-x}}\right)^{-1/2} \cdot \left[3 + \frac{1}{2} \cdot \left(2 + \sqrt{1-x}\right)^{-1/2} \cdot \left(\frac{1}{2}(1-x)^{-1/2}(-1)\right)\right]$$

b) $$g'(x) = -1 \cdot \left(\frac{1 + \sin 3x}{3 - 2x}\right)^{-2} \cdot \left[\frac{1 + \sin 3x}{3 - 2x}\right]'$$

$$= -1 \cdot \left(\frac{1 + \sin 3x}{3 - 2x}\right)^{-2} \left[\frac{(3\cos 3x) \cdot (3 - 2x) - (-2) \cdot (1 + \sin 3x)}{(3 - 2x)^2}\right]$$

Question 7

5 Kasım 2020 Perşembe 13:01

a) Suppose f is a differentiable function and $y = x/4 - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point $x = 8$. If $g(x) = (f(x^3))^2$, find an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 2$.
b) If $g''(2) = 0$, find $f''(8)$.

Solution:

a) (f is diff. func.) Tangent line is given by at $x=x_0$ to $y=f(x)$

$$\boxed{y - y_0 = m \cdot (x - x_0)}$$ slope $m = f'(x_0)$

$y - y_0 = f'(x_0)(x - x_0)$ $\quad \boxed{x_0 = 8}$ $\quad y_0 = f(x_0) = f(8)$

$\Rightarrow y - f(8) = f'(8)(x-8) \Rightarrow \boxed{y = f'(8)(x-8) + f(8)}$ is the tanget line to $f(x)$ at $x=8$

from the question we know the tangent line is

is given in the quest.

$$\boxed{y = \frac{x}{4} - 3} = f'(8)(x-8) + f(8)$$

$$f'(8)x - 8f'(8) + f(8)$$

$$\boxed{f'(8) = \frac{1}{4}} \qquad -3 = -8 \cdot \underbrace{f'(8)}_{\frac{1}{4}} + f(8) \Rightarrow \boxed{f(8) = -1}$$

Now, we look for the tangent line to $y = g(x)$ at $x=2$

$\Rightarrow y - y_0 = m \cdot (x - x_0)$ $\quad x_0 = 2 \Rightarrow y_0 = g(x_0) = g(2)$

$m = g'(x_0) = g'(2)$

$\Rightarrow \boxed{y - g(2) = g'(2)(x-2)}$

We need to find the value of $g(2)$ and $g'(2)$

$g(x) = [f(x^3)]^2 \Rightarrow g'(x) = 2 \cdot [f(x^3)] \cdot f'(x^3) \cdot 3x^2$

$g(2) = [f(8)]^2 = 1$ and $g'(2) = 2 \cdot f(8) \cdot f'(8) \cdot 3 \cdot 4$

$g'(2) = 2 \cdot (-1) \cdot \frac{1}{4} \cdot 3 \cdot 4 = -6$

$$y - 1 = -6.(x-2) \Rightarrow \boxed{y = -6x + 13}$$

b) $g''(2) = 0 \Rightarrow f''(8) = ?$

$$g(x) = \left[f(x^3)\right]^2 \Rightarrow g'(x) = 2.f(x^3).f'(x^3).3x^2$$

$(f.g.h)' = f'gh + f.g'.h + f.g.h'$

$$g''(\underset{2}{x}) = 6.\Big[f'(\underset{8}{x^3}).3\underset{4}{x^2}.f'(\underset{8}{x^3}).\underset{4}{x^2} + f(\underset{8}{x^3}).f''(\underset{8}{x^3}).3\underset{4}{x^2}.\underset{4}{x^2} + f(\underset{8}{x^3}).f'(\underset{8}{x^3}).2\underset{2}{x}\Big]$$

$$0 = 6.\left[\frac{1}{4}.3.4.\frac{1}{4}.4 + (-1).f''(8).3.4.4 + (-1).\frac{1}{4}.2.2\right]$$

$\Rightarrow f''(8) \Rightarrow$ you can calculate from above :)