Recitation 03: Continuity

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Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

Topics to be covered: (Oct 26-30)

1.4 Continuity

1.5 The Formal Definition of Limit

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DEFINITION: (Continuity at an interior point)

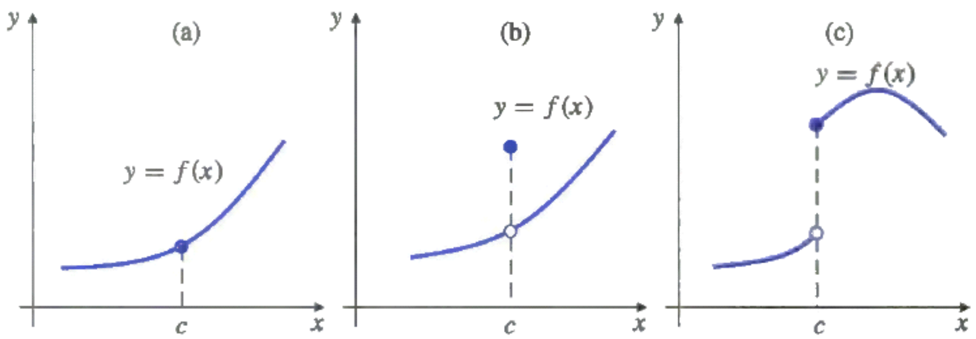
We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

If either $\lim_{x\to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

NOTE:

In graphical terms, f is continuous at an interior point c of its domain if **its graph has no break** in it at the point (c, f (c)) ; in other words, *if you can draw the graph through that point without lifting your pen from the paper*.



DEFINITION: (Right and left continuity)

We say that f is **right continuous** at c if $\lim_{x\to c+} f(x) = f(c)$.

We say that f is **left continuous** at c if $\lim_{x\to c-} f(x) = f(c)$.

THEOREM:

Function *f is continuous* at c **if and only if** it is *both right continuous and left continuous* at c.

DEFINITION: (Continuity at an endpoint)

- We say that f is continuous at a left endpoint c of its domain if it is right continuous there.
- We say that f is continuous at a right endpoint c of its domain if it is left continuous there .

DEFINITION: (Continuity on an interval)

We say that function f is continuous on the interval I if it is continuous at each point of I. In particular, we will say that f is a continuous function if f is continuous at every point of its domain.

NOTE:

The following functions are continuous wherever they are defined:
(a) all polynomials;
(b) all rational functions;
(c) all rational powers ;
(d) the sine, cosine, tangent, secant, cosecant, and cotangent functions
(e) the absolute value function.

THEOREM: (The Max-Min Theorem)

If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then there exist numbers p and q in $[a, b]$ such that for all x in $[a, b]$,

$$f(p) \leq f(x) \leq f(q).$$

Thus f has the absolute minimum value $m = f(p)$, taken on at the point p, and the absolute maximum value $M = f(q)$, taken on at the point q.

THEOREM: (The Intermediate-Value Theorem)

If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.

DEFINITION: (A formal definition of limit)

We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

Question 1

29 Ekim 2020 Perşembe 13:56

(1) Find m so that

$$g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \geq 3 \end{cases}$$

is continuous for all x.

Solution :

• if $x<3$: $g(x) = x - m$ is a poly. func. which is a cont on its domain.

• if $x>3$: $g(x) = 1 - mx$ is a poly. func. which is a cont. on its domain.

• $x=3$: $g(3) = 1-3m$

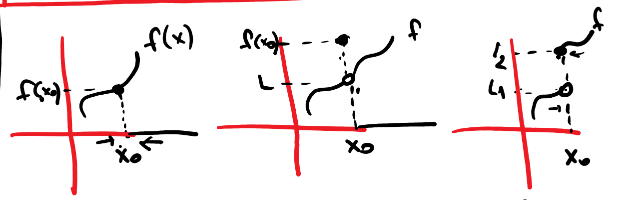
$$\lim_{x\to 3^-} g(x) = \lim_{\substack{x\to 3^- \\ x<3}} x-m = 3-m$$

$$\lim_{x\to 3^+} g(x) = \lim_{\substack{x\to 3^+ \\ x>3}} 1-mx = 1-3m$$

Recall:

f is cont at $x=x_0$ if

- f is defined at $x=x_0$
- $\lim_{x\to x_0} f(x)$ exist!
- $\lim_{x\to x_0} f(x) = f(x_0)$



f is cont. at x_0

f is not cont at x_0

f is not cont. x_0

We know that g is cont on everywhere

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^+} g(x) = g(3)$$

$$3-m = 1-3m = 1-3m$$

$$-2m = -2$$

$$\boxed{m=-1}$$

Question 2

29 Ekim 2020 Perşembe 14:04

Assume that f is a real-valued continuous function such that

$\lim_{x \to 0} f(x) \cos^2\left(\frac{\pi}{x}\right) = 0.$

Find f(0).

Sq. thm does not work why?

$0 \le \cos^2\left(\frac{\pi}{x}\right) \le 1$

$0 \le f(x)\cos^2\left(\frac{\pi}{x}\right) \le f(x)$

lim then lim

Solution:

Since f is a cont func, for any x_0 $\lim_{x \to x_0} f(x) = f(x_0)$

claim: $f(0) = 0$

start with an assumption.

We want to prove the claim above. Assume that $f(0) \neq 0$

$f(0) = L \neq 0$

$$\lim_{x \to 0} \cos^2\left(\frac{\pi}{x}\right) = \lim_{x \to 0} \frac{f(x) \cdot \cos^2\left(\frac{\pi}{x}\right)}{f(x)} = \frac{\lim_{x \to 0} f(x)\cos^2\left(\frac{\pi}{x}\right)}{\lim_{x \to 0} f(x)} = \frac{0}{L} = 0$$

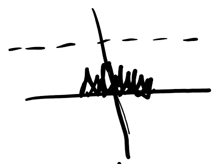
Since we know f is a cont. func. $\lim_{x \to 0} f(x) = f(0) = L$ $(L \neq 0)$

we have $\lim_{x \to 0} f(x)\cos^2\left(\frac{\pi}{x}\right) = 0$ (given in the question)

⇒ Using the assumption, we have $\lim_{x \to 0} \cos^2\left(\frac{\pi}{x}\right) = 0$

On the hand, $\lim_{x \to 0} \cos^2\left(\frac{\pi}{x}\right)$ does not exist!

check the graph of func.



We have a contradiction ⇓ Assumption is wrong!!

Therefore $f(0) = 0$

Question 3

29 Ekim 2020 Perşembe 14:00

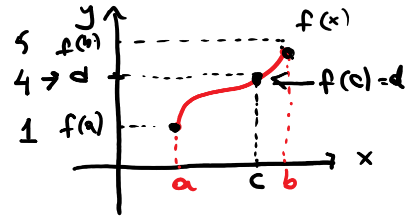
(2) Show that there is some a with $0 < a < 2$ such that $\boxed{a^2 + \cos(\pi a) = 4}$.

$f(a) = 4$

Recall: (IVT → continuity)

f is a cont. on $[a,b]$ and (for any) d is between $f(a)$ and $f(b)$

$\Rightarrow$ $\exists$ at least one $c \in (a,b)$ s.t. $f(c) = d$.



Solution:

Define $f(x) = x^2 + \cos(\pi x)$. f is cont on $\mathbb{R}$. Since it is addition of a polyn. and a trigonometric fun. (polyn. and trig. are cont on $\mathbb{R}$)

f is cont on $[0,2]$ also.

$f(0) = 1$ and $f(2) = 4 + 1 = 5$. We have the following:

4 is between $f(0) = 1$ and $f(2) = 5$.

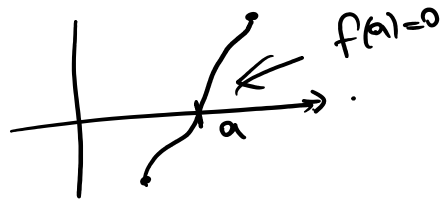
By IVT; $\exists\, a \in (0,2)$ s.t $f(a) = 4 \Leftrightarrow a^2 + \cos(\pi a) = 4$

$a^2 + \cos(\pi a) - 4$

2nd way:

Define $f(x) = x^2 + \cos(\pi x) - 4$. We want to prove

$\exists\, a \in (0,2)$ s.t. $f(a) = 0$ $(d = 0)$



$f(0) = -3 < 0$

$f(2) = 1 > 0$

Question 4

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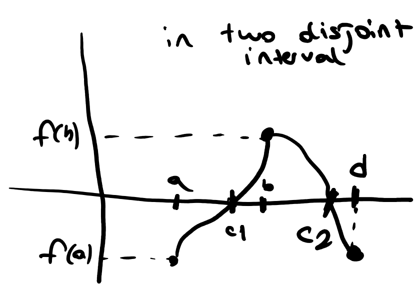
Show that the following equation has at least two solutions.

$$\cos(x) = x^2 - 1$$

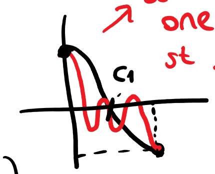


Solution:

Define $f(x) = \cos(x) - x^2 + 1$. We want to show that f has at least two roots.

f is cont. on $\mathbb{R}$ since it is an addition of two func. which are polyn. and trigonometric. (funcs. are cont on $\mathbb{R}$)

also f is cont on $[0, \frac{\pi}{2}]$, $[-\frac{\pi}{2}, 0]$



$f(-\frac{\pi}{2}) = -\frac{\pi^2}{4} + 1 < 0$

$f(0) = 2 > 0$

$f(\frac{\pi}{2}) = -\frac{\pi^2}{4} + 1 < 0$

By IVT, $\exists c_2 \in (-\frac{\pi}{2}, 0)$ s.t. $f(c_2) = 0$

By IVT; $\exists c_1 \in (0, \frac{\pi}{2})$ s.t. $f(c_1) = 0$

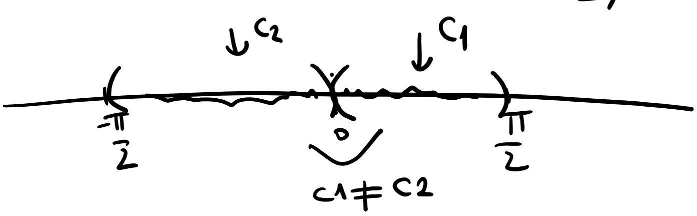
why?

$\frac{\pi}{2} > \frac{3}{2}$

$\frac{\pi^2}{4} > \frac{9}{4}$

$-\frac{\pi^2}{4} + 1 < -\frac{9}{4} + 1 = -\frac{5}{4}$

$\Rightarrow$



Therefore, f has at least two roots!!

(Since the intervals are disjoints.)

Question 5

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(4) Use the formal definition of the limit to verify the following:

(a) $\lim_{x\to c}(ax+b) = ac+b$ for any $a, b, c \in \mathbb{R}$.

(b) $\lim_{x\to 2} \frac{x-2}{1+x^2} = 0$

(c) $\lim_{x\to 3} \sqrt{2x+3} = 3$

(ε-δ defn.)

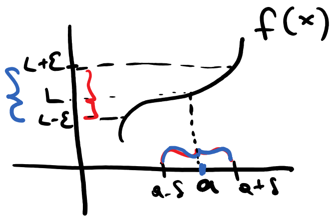
Recall : (formal defn of limit)

$\lim_{x\to a} f(x) = L$

$\Updownarrow$

for any $\varepsilon > 0$; $\exists \delta > 0$ s.t.

$0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$



Solution :

a) $\lim_{x\to c}(ax+b) = a.c+b$ for $a,b,c \in \mathbb{R}$

We want to prove that using formal defn:

for any $\varepsilon > 0$; choose $\delta = \frac{\varepsilon}{|a|}$ $(a \neq 0)$ s.t.

$0 < |x-c| < \delta$ implies $|ax+b-(ac+b)| < \varepsilon$

Let's consider

$|ax+b-(ac+b)| = |ax-ac| = |a|.|x-c| < |a|.\delta = \varepsilon$

$\frac{\varepsilon}{|a|}$

• if $a=0$: for this case any δ works! Since $|ax+b-(ac+b)| = 0 < \varepsilon$

$= 0 < \varepsilon$.

$\delta = \varepsilon, 2\varepsilon, \frac{\varepsilon}{2} \ldots$; $1, 2$

• if $a \neq 0$: choose $\delta \leq \frac{\varepsilon}{|a|} \Rightarrow |ax+b-(ac+b)| = |a|.|x-c|$

$\leq |a| . \frac{\varepsilon}{|a|} = \varepsilon$.

b) $\lim_{x\to 2} \frac{x-2}{1+x^2} = 0$

for any $\varepsilon > 0$; choose $\delta = \varepsilon$?? $(\varepsilon/2, \varepsilon/3 \cdots)$ s.t.

$0 < |x-2| < \delta$ implies $\left|\frac{x-2}{1+x^2} - 0\right| < \varepsilon$

Let's consider

$\left|\frac{x-2}{1+x^2}\right| = \frac{|x-2|}{|1+x^2|} \leq |x-2| < \delta = \varepsilon$

choose $\delta \leq \varepsilon$ $\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{3} \cdots\right)$

since $x^2 \geq 0$. $1+x^2 \geq 1$

choose $\delta \leq \varepsilon$.

st.

$$\left(\text{since} \begin{cases} x^2 \geq 0. \\ 1+x^2 \geq 1 \\ \dfrac{|x-2|}{1+x^2} \leq |x-2| \end{cases} \right)$$

s.t.

c) $\lim\limits_{x \to 3} \sqrt{2x+3} = 3.$

ε-δ defn

$$\begin{cases} \text{for any } \varepsilon > 0; \text{ choose } \delta \leq \dfrac{3\varepsilon}{2} \ ?? \text{ s.t.} \\ \boxed{0 < |x-3| < \delta} \text{ implies } \left| \sqrt{2x+3} - 3 \right| < \varepsilon \end{cases}$$

Let's consider

$$\left| \sqrt{2x+3} - 3 \right| = \frac{\left|\sqrt{2x+3} - 3\right| \cdot \left(\sqrt{2x+3} + 3\right)}{\left(\sqrt{2x+3} + 3\right)} = \frac{|\overbrace{2x+3-9}^{2x-6}|}{\sqrt{2x+3} + 3}$$

$$= \frac{2 \cdot |x-3|}{\sqrt{2x+3} + 3} \leq \frac{2}{3} \cdot |x-3| < \frac{2}{3} \cdot \delta \leq \frac{2}{3} \cdot \frac{3\varepsilon}{2} = \varepsilon$$

$$\left(\begin{array}{l} \sqrt{2x+3} \geq 0 \\ \sqrt{2x+3} + 3 \geq 3 \\ \dfrac{2 \cdot |x-3|}{\sqrt{2x+3} + 3} \leq \dfrac{2 \cdot |x-3|}{3} \end{array} \right)$$

Choose: $\delta \leq \dfrac{3\varepsilon}{2}$

$< \varepsilon$