From last week- Question 4

21 Ekim 2020 Çarşamba 16:23

4. Let $f(x) = x - \lfloor x \rfloor$

a. Sketch the graph of f.

b. If n is an integer, evaluate

(i) $\lim_{x\to n^-} f(x)$

(ii) $\lim_{x\to n^+} f(x)$

c. For what values of a does $\lim_{x\to a} f(x)$ exist?

(*) the func. gives the greates integer less than or equal to x

$\lfloor x \rfloor \rightarrow$ the greatest integer func.

(*) $a \in \mathbb{Z} \Rightarrow a \leq x \leq a+1$

$\lfloor x \rfloor = a.$

ex: $\lfloor 2.1 \rfloor = 2$

$\lfloor 4 \rfloor = 4$

$\lfloor -1.2 \rfloor = -2$

Solution:

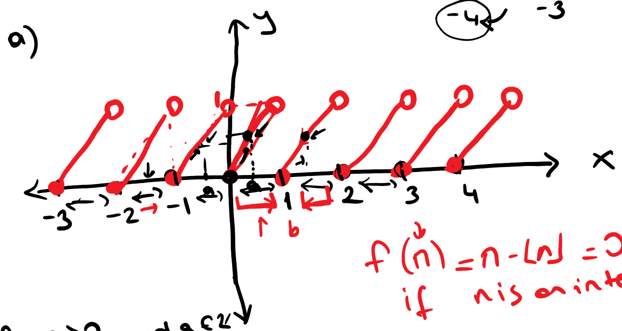
$$f(x) = x - \lfloor x \rfloor = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ x - a & \text{if } a < x < a+1 \text{ where } a \in \mathbb{Z} \end{cases}$$

$f(2) = 2 - \lfloor 2 \rfloor = 2 - 2 = 0$

$f(2.5) = 2.5 - \lfloor 2.5 \rfloor = 0.5$

Ex: $f(-3.4) = -3.4 - \lfloor -3.4 \rfloor = -3.4 + 4 = 0.6$

a)



$f(x) = x - \lfloor x \rfloor$

$1 \leq x < 2 \Rightarrow 0 \leq f(x) < 1$

$\lfloor x \rfloor = 1$

$f(x) = x - 1$

$-3 \leq x < -2 \Rightarrow f(x) = x + 3$

$\lfloor x \rfloor = -3$

$0 \leq x + 3 < 1$

$f(n) = n - \lfloor n \rfloor = 0$ if n is an integer.

(*) if $a > 0$ and $a \in \mathbb{Z}$

$a < x < a+1$

$f(x) = x - \lfloor x \rfloor \Rightarrow f(x) = x - a$

$0 < x - a < 1$

(*) if x is an integer

$f(x) = 0$

(*) if $a < 0$ and $a \in \mathbb{Z}$:

$a < x < a+1$

$f(x) = x - \lfloor x \rfloor \Rightarrow f(x) = x - a \Rightarrow 0 < x - a < 1$

b) i) $\lim\limits_{\substack{x \to n^- \\ x < n}} f(x) = 1$ (from graph)

(n is an integer)

$\lim\limits_{\substack{x \to n^- \\ x < n}} x - \lfloor x \rfloor = n - (n-1) = 1$

$n-1$ n

ii) $\lim\limits_{x \to n^+} f(x) = 0$ from graph

$\lim\limits_{\substack{x \to n^+ \\ x > n}} x - \lfloor x \rfloor = n - n = 0$

n $n+1$

c) $\lim\limits_{x \to b} f(x)$ (*) for values of b this limit exist?

from part b: we can say that case 1) if $b \in \mathbb{Z}$ (integer)

... does not exist

from part b: we can say that if $b \in \mathbb{Z}$ (integer)

$$\lim_{x\to b^+} f(x) = 0 \neq \lim_{x\to b^-} f(x) = 1 \Rightarrow \lim_{x\to b} f(x) \text{ does not exist}$$

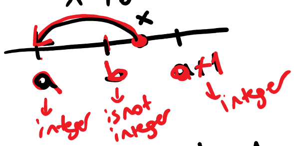
when b is an integer.

case 2)

(*) if b is not an integer: ($\boxed{a < b < a+1}$ where a is an integer)

$$\lim_{x\to b^+} f(x) = \lim_{x\to b^+} x - \lfloor x \rfloor = \boxed{b-a}$$

$x > b$

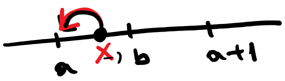


$$\Rightarrow \lim_{x\to b} f(x) = b-a$$

where b is not an integer $a < b < a+1$

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^-} x - \lfloor x \rfloor = \boxed{b-a}$$

$x < b$



Therefore $\lim_{x\to b} f(x)$ exist only when $b \notin \mathbb{Z}$ $(b \in \mathbb{R} - \mathbb{Z})$

From last week- Question 8

21 Ekim 2020 Çarşamba 16:27

8. Evaluate

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2-2x}-2}$$

Solution:

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2-2x}-2} = \lim_{x\to\pm\infty} \frac{1}{\sqrt{\underbrace{x^2}_{|x|}\cdot\left(1-\frac{2}{x}\right)}-2} = \lim_{x\to\mp\infty} \frac{1}{|x|\cdot\underbrace{\sqrt{1-\cancel{\frac{2}{x}}^{\,0}}}_{1}-2} = 0$$

=

From last week- Question 9

21 Ekim 2020 Çarşamba 16:27

9. Evaluate

$$\lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$$

$(x-1)$ $(x+1)$

Solution:

$$\lim_{x\to\infty}\frac{x^2.(x-1)-x^2.(x+1)}{(x-1)(x+1)}=\lim_{x\to\infty}\frac{x^2(x-1-x-1)}{x^2-1}=\lim_{x\to\infty}\frac{x^2.(-2)}{x^2-1}$$

$$\lim_{x\to\infty}\frac{\cancel{x^2}.(-2)}{\cancel{x^2}.(1-\cancel{\frac{1}{x^2}}^{0})}=\frac{-2}{1}=-2$$

Recitation 03: Continuity

21 Ekim 2020 Çarşamba 16:18

Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

Topics to be covered: (Oct 26-30)

1.4 Continuity

1.5 The Formal Definition of Limit

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DEFINITION: (Continuity at an interior point)

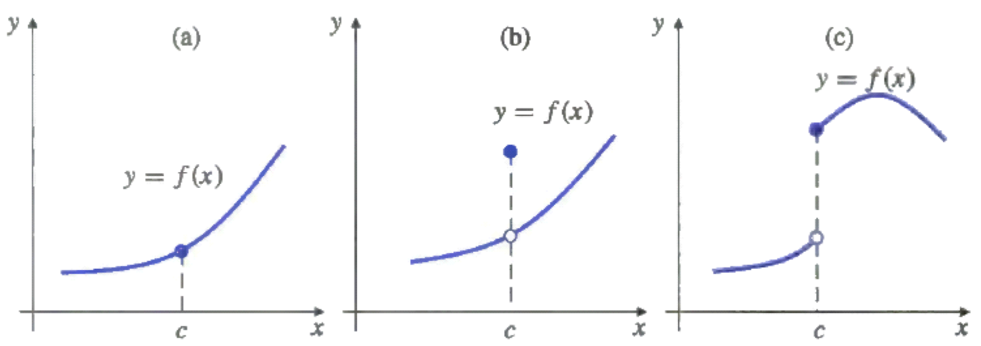
We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

If either $\lim_{x\to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

NOTE:

In graphical terms, f is continuous at an interior point c of its domain if **its graph has no break** in it at the point (c, f (c)) ; in other words, *if you can draw the graph through that point without lifting your pen from the paper*.



DEFINITION: (Right and left continuity)

We say that f is **right continuous** at c if $\lim_{x\to c+} f(x) = f(c)$.

We say that f is **left continuous** at c if $\lim_{x\to c-} f(x) = f(c)$.

THEOREM:

Function *f is continuous* at c **if and only if** it is *both right continuous and left continuous* at c.

DEFINITION: (Continuity at an endpoint)

- We say that f is continuous at a left endpoint c of its domain if it is right continuous there.
- We say that f is continuous at a right endpoint c of its domain if it is left continuous there .

DEFINITION: (Continuity on an interval)

We say that function f is continuous on the interval I if it is continuous at each point of I. In particular, we will say that f is a continuous function if f is continuous at every point of its domain.

NOTE:

The following functions are continuous wherever they are defined:
(a) all polynomials;
(b) all rational functions;
(c) all rational powers ;
(d) the sine, cosine, tangent, secant, cosecant, and cotangent functions
(e) the absolute value function.

THEOREM: (The Max-Min Theorem)

If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then there exist numbers p and q in $[a, b]$ such that for all x in $[a, b]$,

$$f(p) \le f(x) \le f(q).$$

Thus f has the absolute minimum value $m = f(p)$, taken on at the point p, and the absolute maximum value $M = f(q)$, taken on at the point q.

THEOREM: (The Intermediate-Value Theorem)

If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.

DEFINITION: (A formal definition of limit)

We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

Question 1

29 Ekim 2020 Perşembe 13:56

(1) Find m so that

$$g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \geq 3 \end{cases}$$

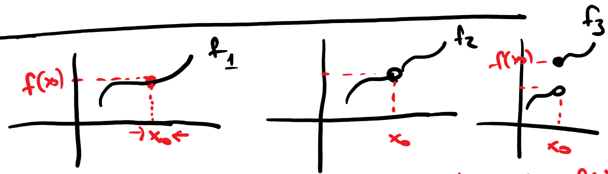
is continuous for all x.

Recall: (continuity)

(*) If f is cont on $x = x_0$

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \left(\begin{array}{l} f(x_0) \\ \text{must be} \\ \text{defined} \end{array} \right)$$

Ex:



continous at x_0

• $\lim_{x \to x_0} f(x)$ exist
• f is undefined at x_0
f is not continous

$\lim f(x)$ does not exist.

Solution:

• if $x < 3$: $g(x) = x - m$ is a poly. func.

Therefore it is cont. on its domain.

• if $x > 3$: $g(x) = 1 - mx$ is a poly func also.

Therefore it is cont. on its domain

• if $x = 3$:

• $g(3) = 1 - 3m$

• $\lim_{x \to 3^-} g(x) = \lim_{\substack{x \to 3^- \\ (x<3)}} (x - m) = 3 - m$

• $\lim_{x \to 3^+} g(x) = \lim_{\substack{x \to 3^+ \\ (x>3)}} 1 - mx = 1 - 3m$

$$\underbrace{\lim_{x \to 3^+} g(x)}_{} = \underbrace{\lim_{x \to 3^-} g(x)}_{} = \underbrace{g(3)}_{}$$

$$\boxed{1 - 3m = 3 - m} = 1 - 3m$$

$\Rightarrow 2m = -2$

$\Rightarrow \boxed{m = -1}$

Question 2

29 Ekim 2020 Perşembe 14:04

Assume that f is a real-valued continuous function such that

$$\lim_{x\to 0} f(x)\cos^2\left(\frac{\pi}{x}\right) = 0.$$

Find f(0).

$$0 \le \cos^2\left(\frac{\pi}{x}\right) \le 1$$

$$0 \le f(x)\cos^2\left(\frac{\pi}{x}\right) \le f(x)$$

$$\Downarrow$$

$$0$$

We can not use squeeze thm.

Solution: for any $x_0 \in \mathbb{R}$

$\lim_{x\to x_0} f(x) = f(x_0)$ Since f is a cont. func.

$\boxed{\text{claim: } f(0)=0}$

We need to prove that. $\underbrace{\text{Assume that } \boxed{f(0)\neq 0}}_{\text{start with assumption.}}$. Therefore

$f(0) = L \quad (L \neq 0)$ since we have $f(0)\neq 0$

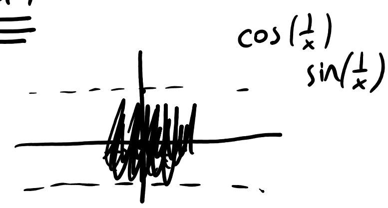
$$\lim_{x\to 0} \cos^2\left(\frac{\pi}{x}\right) = \lim_{x\to 0} \frac{f(x)\cos^2\left(\frac{\pi}{x}\right)}{f(x)} = \frac{\lim_{x\to 0} f(x)\cos^2\left(\frac{\pi}{x}\right)}{\lim_{x\to 0} f(x)} = \frac{0}{L} = 0$$

Since, $\lim_{x\to 0} f(x)\cos^2\left(\frac{\pi}{x}\right) = 0$ and $\lim_{x\to 0} f(x) = f(0) = L \neq 0$,

since f is cont.

comes from assumption

We find that: $\boxed{\lim_{x\to 0} \cos^2\left(\frac{\pi}{x}\right) = 0}$ BUT



$\lim_{x\to 0} \cos^2\left(\frac{\pi}{x}\right)$ DOES NOT EXIST!

please check the graph!

We have a contradiction $\Rightarrow$ Assumption is wrong!

$$\boxed{f(0)=0}$$

why sq. does not work?

① $g(x) \le f(x) \le h(x)$

② $\boxed{\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L}$

we need to have 2 to use sq.

$\Rightarrow \lim_{x\to c} f(x) = L$

Question 3

29 Ekim 2020 Perşembe 14:00

(2) Show that there is some a with $0 < a < 2$ such that $a^2 + \cos(\pi a) = 4$.

$f(a) = 4 = d$

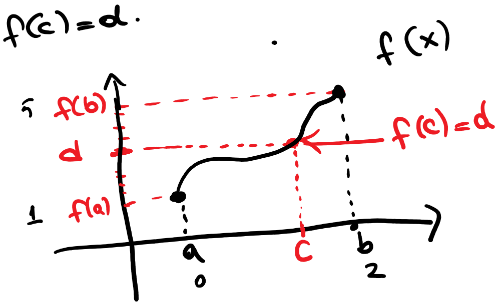
Recall: (IVT → continuity)

If $f(x)$ is a cont on $[a,b]$

d is between $f(a)$ and $f(b)$

then $\exists c \in (a,b)$ s.t

$f(c) = d$.



1st way

Solution:

Define $f(x) = x^2 + \cos(\pi x)$. f is cont on $\mathbb{R}$. Since it is an addition of a polyn. func and trigonometric func. (We know poly and trig. are cont on $\mathbb{R}$.)

It is cont on $[0,2]$ also.

$f(0) = 1$ and $f(2) = 4 + 1 = 5$ (4 is between 1 and 5)

By IVT, $\exists a \in (0,2)$ s.t $f(a) = a^2 + \cos(\pi a) = 4$

2nd way: $f(x) = x^2 + \cos(\pi x) - 4$

↳ func has at least one root.

↳ $f(a) < 0$ $f(b) > 0$ } By IVT $\exists c \in (a,b)$

Question 4

29 Ekim 2020 Perşembe 14:04

Show that the following equation has at least two solutions.

$$\cos(x) = x^2 - 1$$

Solution:

Define $f(x) = \cos(x) - x^2 + 1$.

f is cont. on $\mathbb{R}$. Since it is an addition of poly. and trigonometric func.

It is cont on $\left[-\frac{\pi}{2}, 0\right]$ and $\left[0, \frac{\pi}{2}\right]$ also.

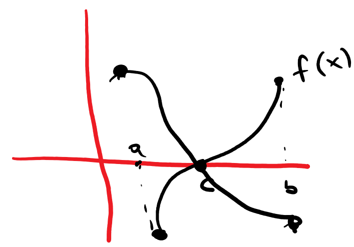
$f\left(-\frac{\pi}{2}\right) = -\frac{\pi^2}{4} + 1 < 0$

$f(0) = 2 > 0$

$f\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4} + 1 < 0$

By IVT $\exists c_2 \in \left(-\frac{\pi}{2}, 0\right)$ s.t $f(c_2) = 0$

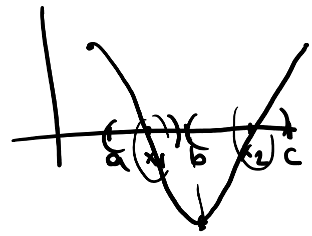
By IVT $\exists c_1 \in \left(0, \frac{\pi}{2}\right)$ s.t $f(c_1) = 0$



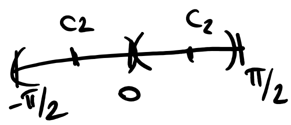
$f(c) = 0$ c is a root of func.

f is cont on $[a,b]$

$f(a) < 0$ $f(b) > 0$



2 disjoint intervals.



Since $\left(-\frac{\pi}{2}, 0\right)$ and $\left(0, \frac{\pi}{2}\right)$ are disjoint, func has at least two roots.

Question 5

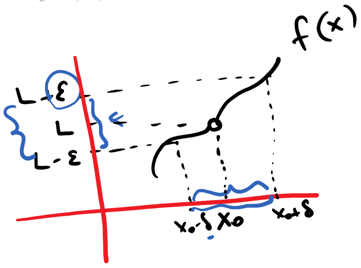
29 Ekim 2020 Perşembe 14:02

(4) Use the formal definition of the limit to verify the following:

(a) $\lim_{x\to c}\overbrace{(ax+b)}^{f(x)} = \overbrace{ac+b}^{L}$ for any $a,b,c\in\mathbb{R}$.

(b) $\lim_{x\to 2}\frac{x-2}{1+x^2}=0$

(c) $\lim_{x\to 3}\sqrt{2x+3}=3$



(ε-δ defn)

Recall : (Formal defn of limit)

For a given $\varepsilon>0$. There exist $\exists\delta>0$ s.t (← you need choose a δ. satisfy)

$$\boxed{0<|x-x_0|<\delta \text{ implies that } |f(x)-L|<\varepsilon .}$$

$$\Updownarrow$$

$$\boxed{\lim_{x\to x_0} f(x) = L}$$

Solution :

a) $\lim_{x\to c} ax+b = ac+b$ for any $a,b,c\in\mathbb{R}$.

for any $\varepsilon>0$. choose $\delta=\frac{\varepsilon}{|a|}$ $(a\neq 0)$ s.t.

$0<|x-c|<\delta$ implies $|ax+b-(ac+b)|<\varepsilon$.

Let's consider

$$|ax+b-(ac+b)| = |ax-ac| = |a.(x-c)| = \overbrace{|a|.\underbrace{|x-c|}_{<\frac{\varepsilon}{|a|}}}^{<\varepsilon}$$

- if $a=0$: every δ works. Since $|ax+b-(ac+b)|=0<\varepsilon$
- if $a\neq 0$: choose $\delta=\frac{\varepsilon}{|a|}$

b) $\lim_{x\to 2}\frac{x-2}{1+x^2}=0$

for a given $\varepsilon>0$, $\exists\delta>0$
s.t
$0<|x-x_0|<\delta$ implies
$|f(x)-L|<\varepsilon$

For any $\varepsilon>0$, choose $\delta=\varepsilon$ $(\frac{\varepsilon}{2},\frac{\varepsilon}{3}$ s.t

$0<|x-2|<\delta$ implies $\left|\frac{x-2}{1+x^2}-0\right|<\varepsilon$

Let's consider

$x^2\geq 0$
$x^2+1\geq 1$

Let's consider

$$\left|\frac{x-2}{1+x^2}\right| = \frac{|x-2|}{x^2+1} \leq |x-2| < \delta \leq \varepsilon$$

$$\left(\begin{array}{l} x^2 \geq 0 \\ x^2+1 \geq 1 \\ \frac{1}{x^2+1} \leq 1 \end{array}\right)$$

$$\frac{|x-2|}{x^2+1} \leq |x-2|$$

Choose $\delta \leq \varepsilon$

c) $\lim\limits_{x \to 3} \sqrt{2x+3} = 3$

for any $\varepsilon > 0$; choose $\delta = \frac{3\varepsilon}{2}$ s.t

$0 < |x-3| < \delta$ implies $\left|\sqrt{2x+3} - 3\right| < \varepsilon$

Let's consider

$$\left|\sqrt{2x+3} - 3\right| = \frac{\left|(\sqrt{2x+3}-3)(\sqrt{2x+3}+3)\right|}{\left|\sqrt{2x+3}+3\right|} = \frac{|\overbrace{2x+3-9}^{2x-6}|}{\sqrt{2x+3}+3}$$

$$= \frac{2 \cdot |x-3|}{\sqrt{2x+3}+3} \leq \frac{2}{3} \cdot |x-3|$$

$$\left(\begin{array}{l} \sqrt{2x+3}+3 \geq 3 \\ \frac{1}{\sqrt{\cdots}+3} \leq \frac{1}{3} \end{array}\right.$$

choose $\delta \leq \frac{3\varepsilon}{2}$ then $\leq \frac{2}{3} \cdot \frac{3\varepsilon}{2} = \varepsilon$