Recitation 02: Limits

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Math 119 - Calculus with Analytic Geometry

Topics to be covered: (Oct. 19-23)
1.2 Limits of Functions
1.3 Limits at Infinity and Infinite Limits

Course webpage: http://ma119.math.metu.edu.tr/

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INTRODUCTION:

Calculus was created to describe how quantities change. It has two basic procedures that are opposites of one another :

•	**differentiation**, for finding the rate of change of a given function, and
•	**integration**, for finding a function having a given rate of change.

Both of these procedures are based on the fundamental concept of the **limit** of a function.

DEFINITION: (informal)

If f (x) is defined for all x near a, *except possibly at a itself*, and if we can ensure that f(x) is as close as we want to L by taking x close enough to a, *but not equal to a*, **we say that the function f approaches the limit L as x approaches a**, and we write

$$\lim_{x \to a} f(x) = L.$$

DEFINITION: (informal)

If f(x) is defined on some interval (b, a) extending to the left of x = a, and if we can ensure that f (x) is as close as we want to L by taking x to the left of a and close enough to a, then **we say f (x) has left limit L at x = a**, and we write

$$\lim_{x \to a-} f(x) = L.$$

If f (x) is defined on some interval (a , b) extending to the right of x = a, and if we can ensure that f (x) is as close as we want to L by taking x to the right of a and close enough to a, then **we say f(x) has right limit Lat x = a**, and we write

$$\lim_{x \to a+} f(x) = L.$$

THEOREM:

A function *f (x) has limit L at x = a* **if and only if** it *has both left and right limits there and these one-sided limits are both equal to L*:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L.$$

LIMIT RULES:

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and k is a constant, then

1. **Limit of a sum:** $\lim_{x\to a} [f(x) + g(x)] = L + M$
2. **Limit of a difference:** $\lim_{x\to a} [f(x) - g(x)] = L - M$
3. **Limit of a product:** $\lim_{x\to a} f(x)g(x) = LM$
4. **Limit of a multiple:** $\lim_{x\to a} kf(x) = kL$
5. **Limit of a quotient:** $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$.

If m is an integer and n is a positive integer, then

6. **Limit of a power:** $\lim_{x\to a} \left[f(x)\right]^{m/n} = L^{m/n}$, provided $L > 0$ if n is even, and $L \neq 0$ if $m < 0$.

If $f(x) \leq g(x)$ on an interval containing a in its interior, then

7. **Order is preserved:** $L \leq M$

Rules 1–6 are also valid for right limits and left limits. So is Rule 7, under the assumption that $f(x) \leq g(x)$ on an open interval extending in the appropriate direction from a.

Limits of Polynomia ls and Rational Functions:

1. If $P(x)$ is a polynomial and a is any real number, then

$$\lim_{x\to a} P(x) = P(a).$$

2. If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem:

Suppose that **f(x) ≤ g(x) ≤ h(x)** holds for all x in some open interval containing a, except possibly at x = a itself. Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then

$$\lim_{x \to a} g(x) = L$$

also. Similar statements hold for left and right limits.

Limits At Infinity and Infinite Limits:

(i) limits at infinity, where x becomes arbitrarily large, positive or negative;

divide the numerator and denominator by the highest power of x appearing in the denominator!

The limits of a rational function at infinity and negative infinity either both fail to exist or both exist and are equal.

DEFINITION: (informal)

If the function f is defined on an interval (a, oo) and if we can ensure that f (x) is as close as we want to the number L by taking x large enough, then we say that **f (x) approaches the limit L as x approaches infinity**, and we write

$$\lim_{x \to \infty} f(x) = L.$$

If f is defined on an interval (-oo, b) and if we can ensure that f (x) is as close as we want to the number M by taking x negative and large enough in absolute value, then we say that **f (x) approaches the limit M as x approaches negative infinity**, and we write

$$\lim_{x \to -\infty} f(x) = M.$$

(ii) infinite limits, which are not really limits at all but provide useful symbolism for describing the behaviour of functions whose values become arbitrarily large, positive or negative.

we said that such a limit does not exist, but we can assign ∞ or -∞ to such limits.

∞ and -∞ are not numbers!

Question 1

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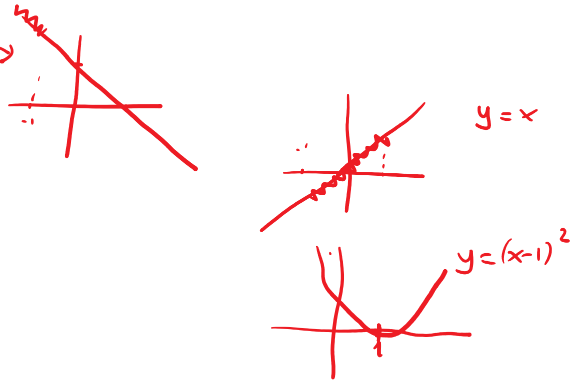
1. Sketch the graph of the following function and evaluate the limits

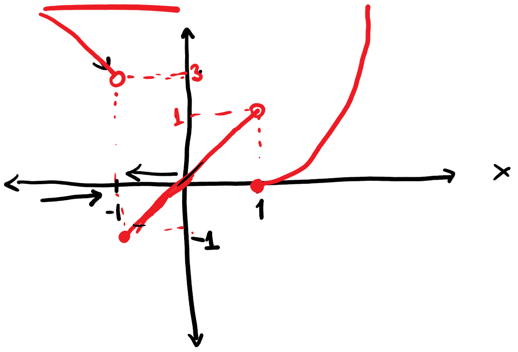
$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ (x-1)^2 & \text{if } x \ge 1 \end{cases}$$

$\to y = 2-x$

a. $\lim_{x\to -1} f(x) = ?$
b. $\lim_{x\to 0} f(x) = ?$
c. $\lim_{x\to 1} f(x) = ?$



Solution :



Recall :

$$\lim_{x\to a} f(x) = L \overset{\text{if and only if}}{\Longleftrightarrow} \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$$

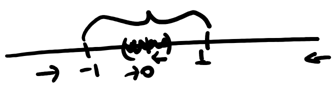
a) $\lim_{\substack{x\to -1^- \\ x<-1}} f(x) = \lim_{x\to -1^-} 2-x = 3$

$\lim_{\substack{x\to -1^+ \\ x>-1}} f(x) = \lim_{x\to -1^+} x = -1$

Since $3 \neq -1$, $\lim_{x\to -1} f(x)$ does not exist!

b) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$

(Since the func is the same both on the right or left side of the point.)



c) $\lim_{\substack{x\to 1^+ \\ x>1}} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0$

$\lim_{\substack{x\to 1^- \\ x<1}} f(x) = \lim_{x\to 1^-} x = 1$

Since $0 \neq 1$, $\lim_{x\to 1} f(x)$ does not exist!

Question 2

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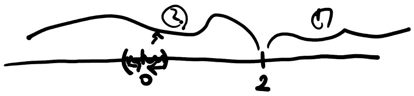
2. Let $f(x) = \frac{|x-2|}{x-2}$.

   a. $\lim_{x\to 0} f(x) = ?$

   b. $\lim_{x\to 2} f(x) = ?$

$$f(x) = \begin{cases} \frac{x-2}{x-2} = 1, & \text{if } x > 2 \\ \frac{-(x-2)}{x-2} = -1 & \text{if } x < 2 \end{cases}$$

(as a note, func is undfn. at $x = 2$!)



Solution:

a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{-\cancel{(x-2)}}{\cancel{x-2}} = -1$

b) $\lim_{\substack{x\to 2^+ \\ x>2 \\ (x\neq 2)}} \frac{\cancel{x-2}}{\cancel{x-2}} = 1$ $\qquad \lim_{\substack{x\to 2^- \\ x<2 \\ (x\neq 2)}} \frac{-\cancel{(x-2)}}{\cancel{x-2}} = -1$

Since $1 \neq -1$, $\lim_{x\to 2} f(x)$ does not exist!

Question 3

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$|x|$ $f(x) = \{$ if $x>a$, $x<a$

3. Evaluate the limit, if it exists

a. $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

b. $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$

c. $\lim_{t\to 0} \frac{(3+t)^{-1}-3^{-1}}{t}$

d. $\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

Recall:

* $a^3-b^3=(a-b)(a^2+ab+b^2)$
* $a^2-b^2=(a-b)(a+b)$

Solution:

a) $$\lim_{x\to 1}\frac{x^3-1}{x^2-1} = \lim_{\substack{x\to 1\\(x\neq 1)}}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1}\frac{x^2+x+1}{x+1} = \frac{3}{2}$$

b) $$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \lim_{\substack{h\to 0\\(h\neq 0)}}\frac{\overbrace{1+h-1}^{\cancel{h}}}{\cancel{h}\cdot(\sqrt{1+h}+1)} = \lim_{h\to 0}\frac{1}{1+\sqrt{1+h}} = \frac{1}{2}$$

c) $$\lim_{t\to 0}\frac{(3+t)^{-1}-3^{-1}}{t} = \lim_{t\to 0}\frac{\frac{1}{3+t}-\frac{1}{3}}{t} = \lim_{\substack{t\to 0\\(t\neq 0)}}\frac{\overbrace{3-(3+t)}^{-\cancel{t}}}{3.(3+t)\cancel{t}}$$

$$= \lim_{t\to 0}\frac{-1}{3.(3+t)} = \frac{-1}{9}$$

Recall
$$\boxed{a^2-b^2=(a-b)(a+b)}$$
$a^3-b^3=(a-b)(a^2+ab+b^2)$

d) $$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}\left(\cdot\frac{\sqrt{x}+x^2}{\sqrt{x}+x^2}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)$$

$$= \lim_{x\to 1}\frac{\overbrace{x-x^4}^{x.(1-x^3)}}{1-x}\cdot\frac{1+\sqrt{x}}{\sqrt{x}+x^2} = \lim_{\substack{x\to 1\\(x\neq 1)}}\frac{x.\cancel{(1-x)}(1+x+x^2)}{\cancel{1-x}}\cdot\left(\frac{\sqrt{x}+1}{\sqrt{x}+x^2}\right)$$

$$= \lim_{x\to 1}\left[x.\frac{\sqrt{x}+1}{\sqrt{x}+x^2}.(1+x+x^2)\right] = \frac{6}{2} = 3$$

Question 4

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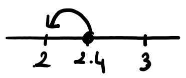
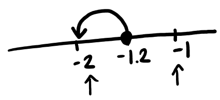
4. Let $f(x) = x - \lfloor x \rfloor$
   a. Sketch the graph of f.
   b. If n is an integer, evaluate
      (i) $\lim_{x\to n^-} f(x)$
      (ii) $\lim_{x\to n^+} f(x)$
   c. For what values of a does $\lim_{x\to a} f(x)$ exist?

(*) Note: $\lfloor x \rfloor$: the greatest integer func.
it takes the greatest int. less than or equal to x.

(*) if $a \le x < a+1$ where a is an integer ($a \in \mathbb{Z}$)
$\lfloor x \rfloor = a$.

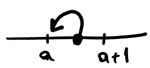
Ex: $\lfloor 2.4 \rfloor = 2$ $\quad \lfloor 3 \rfloor = 3$ $\quad \lfloor -1.2 \rfloor = -2$

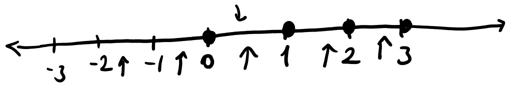




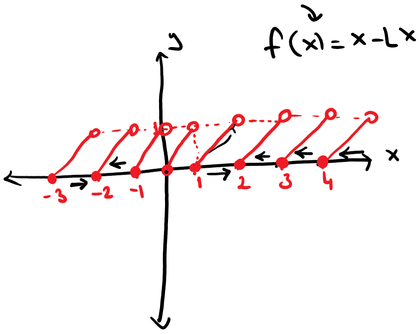
a) $f(x) = x - \lfloor x \rfloor$ → the greatest integer less than or equal x

$$f(x) = \begin{cases} x - x = 0 & \text{if } x \in \mathbb{Z} \text{ (an integer)} \\ x - a & \text{if } x \notin \mathbb{Z} \text{ and } a < x < a+1 \text{ where } a \in \mathbb{Z} \end{cases}$$





$f(x) = x - \lfloor x \rfloor$



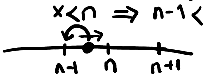
$$f(x) = \begin{cases} 0 & , \text{ if } x \in \mathbb{Z} \\ x - a & , \text{ if } x \notin \mathbb{Z} \text{ and } a < x < a+1 \text{ where } a \in \mathbb{Z} \end{cases}$$

$a < x < a+1$
$0 < x - a < 1 \Rightarrow 0 < f(x) < 1$

b) i) $\lim_{x\to n^-} f(x) = \lim_{x\to n^-} x - \lfloor x \rfloor = n - (n-1) = 1$

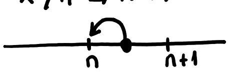
(n is an integer)

$x < n \Rightarrow n-1 < x < n$



ii) $\lim_{x\to n^+} f(x) = \lim_{x\to n^+} x - \lfloor x \rfloor = n - n = 0$

$x > n \Rightarrow n+1 > x > n$



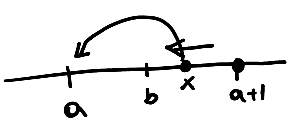
$1 \ne 0$, $\lim_{x\to n} f(x)$ does not exist!

c) $\lim_{x\to b} f(x)$ exist for what values of b?

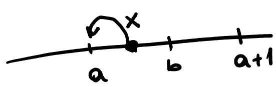
from part b) (*) if $b \in \mathbb{Z} \Rightarrow \lim_{x\to b} f(x)$ does not exist!

(*) if $b \notin \mathbb{Z}$, ($a < b < a+1$ where a is an integer)

$\lim_{x\to b^+} f(x) = \lim_{\substack{x\to b^+ \\ x > b}} x - \lfloor x \rfloor = b - a$



$\lim_{x\to b^-} f(x) = \lim_{\substack{x\to b^- \\ x < b}} x - \lfloor x \rfloor = b - a$



$\Rightarrow \lim f(x) = b - a$ when $b \notin \mathbb{Z}$ and $a < b < a+1$ where a is an integer.

$$\lim_{x<b} \ldots$$

$\Rightarrow \lim_{x \to b} f(x) = b-a$ when $b \notin \mathbb{Z}$ and $a < b < a+1$ where a is an integer.

$\lim_{x \to b} f(x)$ exist only $b \notin \mathbb{Z}$!!

" when $b \in \mathbb{R} - \mathbb{Z}$

Question 5

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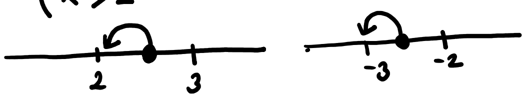
5. If $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$. Show that $\lim_{x\to 2} f(x)$ exist but is not equal to $f(2)$.

Solution:

Let's calculate $f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 - 2 = 0$

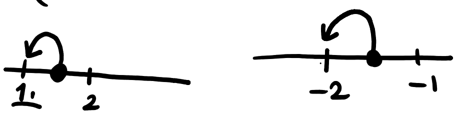
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \lfloor x \rfloor + \lfloor -x \rfloor = 2 + (-3) = -1$$

$(x > 2 \Rightarrow -x < -2)$



$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \lfloor x \rfloor + \lfloor -x \rfloor = 1 + (-2) = -1$$

$(x < 2 \Rightarrow -x > -2)$



Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = -1$, we have

$$\lim_{x\to 2} f(x) = -1. \neq 0 = f(2)$$

Question 6

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6. Evaluate

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$$

Solution:

$$\left[\frac{0}{0}\right]$$

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}$$

$$= \lim_{\substack{x\to 2 \\ x\neq 2}} \frac{\overbrace{6-x-4}^{\cancel{2-x}}}{\underbrace{3-x-1}_{\cancel{2-x}}} \cdot \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{2}{4} = 1/2$$

Question 7

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7. Prove that

a. $\lim_{x\to 0} x^4 \cos(2/x) = 0$

b. $\lim_{h\to 0^+} \sqrt{h}e^{\sin(\pi/h)} = 0$

Recall : (Squeze thm.)

$$\begin{cases} f(x) \le g(x) \le h(x) \quad \text{and} \\ \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \end{cases}$$

$$\Rightarrow \lim_{x\to a} g(x) = L$$

Solution :

a) $-1 \le \cos\left(\frac{2}{x}\right) \le 1$ for all $x \neq 0$

$$\underbrace{-x^4}_{f(x)} \le \underbrace{x^4 \cdot \cos\left(\frac{2}{x}\right)}_{g(x)} \le \underbrace{x^4}_{h(x)}$$

$$\lim_{x\to 0} \underbrace{-x^4}_{f(x)} = 0 = \lim_{x\to 0} \underbrace{x^4}_{h(x)}$$

$\Rightarrow$ By squeze thm,

$$\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

b) show steps

$$\begin{cases} -1 \le \sin\left(\frac{\pi}{h}\right) \le 1 \quad \text{for all } h \neq 0 \\ e^{-1} \le e^{\sin\left(\frac{\pi}{h}\right)} \le e^{1} \quad \text{since exp func is an increasing func.} \\ \frac{\sqrt{h}}{e} \le \sqrt{h} \cdot e^{\sin\left(\frac{\pi}{h}\right)} \le \sqrt{h} \cdot e \end{cases}$$

$x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$ increasing func

$x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$ decreasing func.

$$\lim_{h\to 0^+} \frac{\sqrt{h}}{e} = 0 = \lim_{h\to 0^+} \sqrt{h} \cdot e$$

show

by the sq. thm we have $\lim_{h\to 0^+} \sqrt{h} \cdot \sin\left(\frac{\pi}{h}\right) = 0$

Question 8

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Note: $\infty, -\infty$ are not real numbers!

8. Evaluate

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2-2x}-2}$$

Solution:

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2-2x}-2} = \lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2\cdot\left(1-\frac{2}{x}\right)}-2} = \lim_{x\to\pm\infty} \frac{1}{|x|\sqrt{1-\frac{2}{x}}-2}$$

$\to 0$

1

$= 0$

Question 9

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9. Evaluate

$$\lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$$

$(x-1)$ $(x+1)$

Solution:

$$\lim_{x\to\infty}\frac{x^2.(x-1)-x^2.(x+1)}{(x+1)(x-1)} = \lim_{x\to\infty}\frac{x^2.(x-1-x-1)}{x^2-1} = \lim_{x\to\infty}\frac{x^2.(-2)}{x^2-1}$$

$$\lim_{x\to\infty}\frac{\cancel{x^2}.(-2)}{\cancel{x^2}.\left(1-\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{-2}{1-\cancel{\frac{1}{x^2}}} = \frac{-2}{1} = -2$$