Recitation 01: Preliminaries

13 Ekim 2020 Salı 16:34

Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

METU | Mathematics Department

Gamzegül KARAHİSARLI
gamzegul@metu.edu.tr
https://blog.metu.edu.tr/gamzegul/

Topics to be covered: (Oct. 12-16)

0.1 Real Numbers and the Real Line
0.2 Cartesian Coordinates in the Plane
0.3 Graphs of Quadratic Equations
0.4 Functions and Their Graphs
0.5 Combining Functions to Make New functions
0.6 Polynomials and Rational Functions
0.7 The Trigonometric Functions

Question 1

13 Ekim 2020 Salı 16:41

1. Solve the following inequalities.

$$(a)\ \frac{1}{2x+1} \geq 1 - x, \quad (b)\ |x+3| - 2 > 3x, \quad (c)\ \frac{x^3 - x^2 + 4}{x+3} \leq 1$$

Solution :

a) $\dfrac{1}{2x+1} \geqslant 1 - x \Rightarrow \dfrac{1}{2x+1} + x - 1 \geqslant 0 \Rightarrow \dfrac{1 + (2x+1)(x-1)}{2x+1} \geqslant 0$

$\Rightarrow \boxed{\dfrac{2x^2 - x}{2x+1} \geqslant 0}$ Let's make a sign table to find the sign of the fraction.

Note that the fraction is undfn at $x = -1/2$

		-1/2		0		1/2	
x	−		−		+		+
2x−1	−		−		−		+
2x+1	−		+		+		+
fraction	−	undfn.	+		−		+

Solution set is in the interval

$\left(\frac{-1}{2}, 0\right] \cup \left[\frac{1}{2}, \infty\right)$

b) $|x+3| - 2 > 3x$

Note:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

i) if $x > -3$ $\Rightarrow$ $x + 3 - 2 > 3x$
$(x+3 > 0)$
$\Rightarrow 2x < 1$
$\Rightarrow x < \frac{1}{2} \Rightarrow \left(-3, \frac{1}{2}\right)$

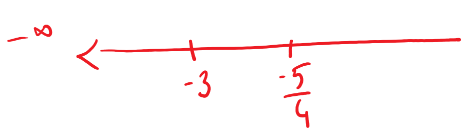
OR

ii) if $x \leq -3$ $\Rightarrow$ $-(x+3) - 2 > 3x$
$(x+3 \leq 0)$
$\Rightarrow 4x < -5$
$\Rightarrow x < \frac{-5}{4} \Rightarrow (-\infty, -3]$

$\Rightarrow$ Solution set in the interval

$\left(-3, \frac{1}{2}\right) \cup (-\infty, -3]$

$= \left(-\infty, \frac{1}{2}\right)$



$-3 \qquad \frac{-7}{4}$

c) $\dfrac{x^3-x^2+4}{x+3} \leq 1 \quad -1 \quad -1 \quad \Rightarrow \dfrac{x^3-x^2+4-x-3}{x+3} \leq 0 \Rightarrow \dfrac{x^2\cdot(x-1)-1\cdot(x-1)}{x+3} \leq 0$

$\Rightarrow \dfrac{(x-1)\overbrace{(x^2-1)}^{(x-1)(x+1)}}{x+3} \leq 0 \Rightarrow \dfrac{(x-1)^2\cdot(x+1)}{x+3} \leq 0$

To find the sign of fraction let's make a sign table.

Note that fraction is undefined at $x=-3$!

		-3		-1		1	
$(x-1)^2$	$+$		$+$		$+$	•	$+$
$x+1$	$-$		$-$	•	$+$		$+$
$x+3$	$-$	•	$+$		$+$		$+$
fraction	$+$	und.	$-$		$+$		$+$

Solution set is in the interval

$(-3,-1] \cup \{1\}$

Question 2

13 Ekim 2020 Salı 16:42

(x_0, y_0) (x_1, y_1)

2. Write an equation for the line through the points $(-1, 5)$ and $(0,3)$.

Solution:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{3-5}{0-(-1)} = -2$$

We can write the eqn. for the line since we have slope and point.

$$y - y_0 = m.(x - x_0)$$

$$y - 5 = -2.(x+1) \Rightarrow \boxed{y = -2x + 3}$$

Note: point-slope eqn.

$$y - y_0 = m.(x - x_0)$$

$(x_0, y_0) \rightarrow$ point

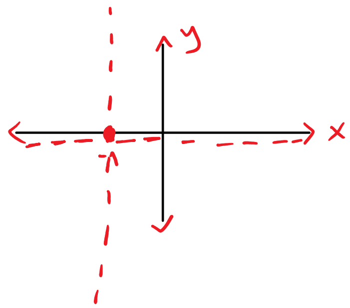
$m \rightarrow$ slope.

Question 3

13 Ekim 2020 Salı 16:42

3. Find the equation for the (a) vertical line and (b) the horizontal line through the point $(-1, 0)$.

Solution:



a) $x=-1$ is the vertical line.

b) $y=0$ is the horizontal line.

Question 4

13 Ekim 2020 Salı 16:43

L2 → slope of line M2.

4. Find the equation for the line through $\boxed{P(-1,3)}$ that is perpendicular to the line $y+x+2=0$. Find the x and y-intercepts of this line.

Solution:

$L_1: y=-x-2 \Rightarrow m_1=-1$

We know that $L_1 \perp L_2 \Rightarrow \overset{-1}{m_1} \cdot m_2 = -1$

$\Rightarrow$ we get $\boxed{m_2 = 1}$

slope point eqn for the line:

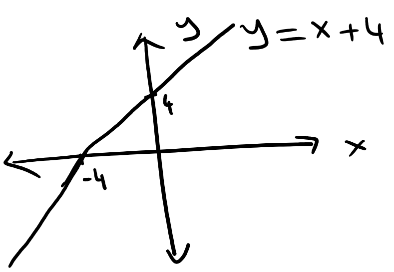
$y - y_0 = m.(x - x_0) \Rightarrow y-3 = 1.(x+1) \Rightarrow \boxed{y = x+4}$

Note: L_1 and L_2: lines

m_1 and m_2: slopes

① $L_1 \perp L_2 \Rightarrow m_1 . m_2 = -1$

② $L_1 // L_2 \Rightarrow m_1 = m_2$



x-intercept : -4

y-intercept : 4

Question 5

13 Ekim 2020 Salı 16:43

5. Describe and sketch the regions defined by the followings

(a) $x^2 + y^2 - 2x - 4y \leq 4$, (b) $x^2 + y^2 \leq 4$, $x^2 + y^2 > 2y$, (c) $x^2 + y^2 > 2y$, $y > 1 + x$

(d) $y > (x-1)^2 + 2$, $y < 2x$, (e) $4x^2 + (y-2)^2 \leq 4$

Solution :

a) $x^2 + y^2 - 2x - 4y \leq 4$

$$\underbrace{x^2 - 2x + 1} + \underbrace{y^2 - 4y + 4} \leq 9$$

$$(x-1)^2 + (y-2)^2 \leq 9$$

is the circle and inside of the circle. (is called a disk)

centre $(1,2)$ $r = 3$

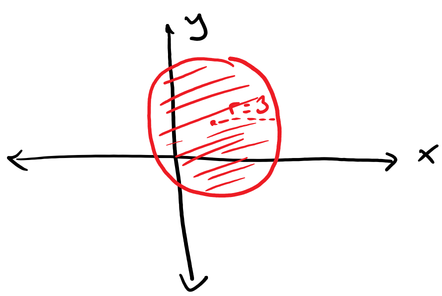
Note :

$(x-a)^2 + (y-b)^2 = r^2$

is defined the circle with

centre (a,b)

radius r.



b) $x^2 + y^2 \leq 4$, $x^2 + y^2 > 2y$

$\Downarrow$

is disk with

centre $(0,0)$

$r = 2$

$\Downarrow$

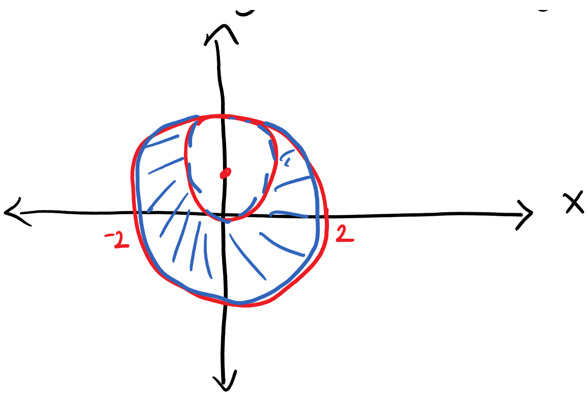
$x^2 + y^2 - 2y + 1 > 1$

$x^2 + (y-1)^2 > 1$ (without boundries)

the outside of the circle

with centre $(0,1)$

$r = 1$



$r = 1$

Since . . . .

Because of

$0 < 1+0 \Rightarrow$ the origin doesn't satisfy ineq.

c) $x^2+y^2 > 2y$, $y > 1+x$

intersection

($y = 1+x$ is a line)

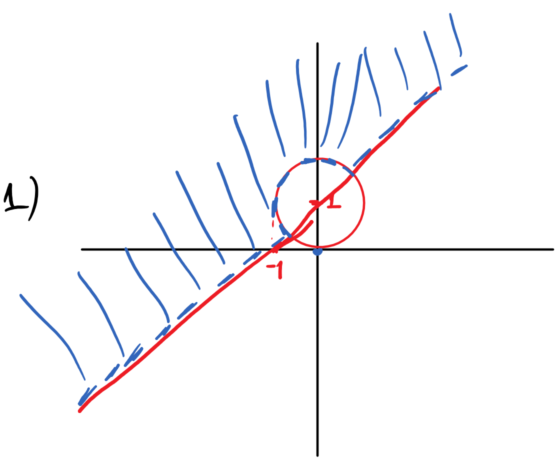
$x^2+y^2-2y+1 > 1$

$x^2+(y-1)^2 > 1$

$\Downarrow$

the outside of the circle centre $(0,1)$ (without boundaries)

$r = 1$



(d) and (e) are exercise for you!

Question 6

13 Ekim 2020 Salı 16:44

6. Find the points of intersection of the pairs of curves

(a) $y = x^2 + 3$, $y = 3x + 1$ (b) $2x^2 + 2y^2 = 5$, $xy = 1$

Solution:

a) $y = x^2+3$, $y = 3x+1$. To find the intersections of the curves, we need to solve together:

$$x^2+3 = 3x+1 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x-2)(x-1) = 0 \Rightarrow \begin{array}{l} x_1 = 2 \Rightarrow y_1 = 7 \\ x_2 = 1 \Rightarrow y_2 = 4 \end{array}$$

$$\begin{array}{cc} x & -2 \\ x & -1 \end{array}$$

The intersection points $(2, 7)$ $(1, 4)$

b) $2x^2 + 2y^2 = 5$, $xy = 1$

$ax^2 + bx + c = 0$

$x_{1,2} = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

Let's solve together:

$xy = 1$ (Observe that; $x = 0$ OR $y = 0$ don't satisfy the eqn)

$\Rightarrow \boxed{y = \frac{1}{x}}$ $(x \neq 0)$

$$2x^2 + 2 \cdot \frac{1}{x^2} = 5 \Rightarrow \frac{2x^4 + 2 - 5x^2}{x^2} = 0 \Rightarrow 2x^4 - 5x^2 + 2 = 0$$

$$\begin{array}{cc} 2x^2 & -1 \\ x^2 & -2 \end{array}$$

$$\Rightarrow (2x^2 - 1)(x^2 - 2) = 0 \Rightarrow x_{1,2} = \mp \frac{1}{\sqrt{2}} \text{ and } x_{3,4} = \mp \sqrt{2}$$

Since we have $y = \frac{1}{x}$ we can calculate $y_{1,2,3,4}$

$$\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right) \quad \left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right) \quad \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) \quad \left(-\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$$

are the intersection points.

Question 7

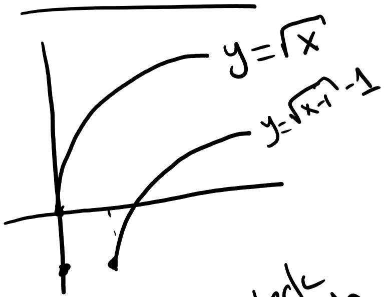
13 Ekim 2020 Salı 16:44

- $y = x$
- $y = \sqrt{x}$
- $y = \sin x$
- $y = |x|$
- $y = \frac{1}{x}$
- $y = x^2$
- $y = \cos x$

7. Write an equation of the graph obtained by shifting the graph of $y = \sqrt{x}$

(a) down 1, right 1 (b) down 2, left 4 (c) up 2, left 1 (d) up 1, right 1

Solution :



a) $y = \sqrt{x-1} - 1$

b) $y = \sqrt{x+4} - 2$

c) $y = \sqrt{x+1} + 2$

d) $y = \sqrt{x-1} + 1$

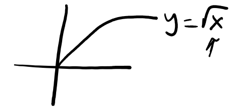
check the graph of the func!!

Note : (Shifting)

$\rightarrow 1$	$g(x-1)$
$\leftarrow 1$	$g(x+1)$
$\uparrow 1$	$g(x)+1$
$\downarrow 1$	$g(x)-1$

Question 8

13 Ekim 2020 Salı 16:45



8. Find the domain and range of each function and sketch their graphs:

(a) $f(x) = \sqrt{x^2 - 1}$ (b) $f(x) = \dfrac{1}{|2 - x|}$ (c) $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$

Solution :

a) $f(x) = \sqrt{x^2 - 1} \Rightarrow x^2 - 1 \geq 0 \Rightarrow x^2 \geq 1 \Rightarrow |x| \geq 1 \Rightarrow x \geq 1$ OR $x \leq -1$

$D_f = \{ x \in \mathbb{R} : x \geq 1 \text{ or } x \leq -1 \}$

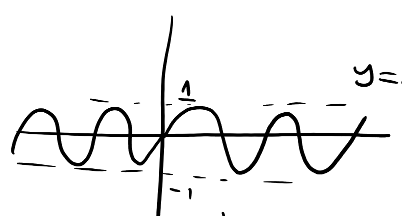
$R_f = \{ y \in \mathbb{R} : y \geq 0 \}$

b) $f(x) = \dfrac{1}{|2 - x|}$ (Observe that $x = 2$ makes the func. undefn.)

$D_f = \{ x \in \mathbb{R} : x \neq 2 \}$

$R_f = \{ y \in \mathbb{R} : y > 0 \}$

c) $f(x) = 1 + \sin\left(x + \frac{\pi}{4}\right)$



$-1 \leq \sin x \leq 1$

$\Downarrow$

$-1 \leq \sin\left(x + \frac{\pi}{4}\right) \leq 1$

$0 \leq \underbrace{1 + \sin\left(x + \frac{\pi}{4}\right)}_{f(x)} \leq 2$

$D_f = \mathbb{R}$

$R_f = [0, 2]$

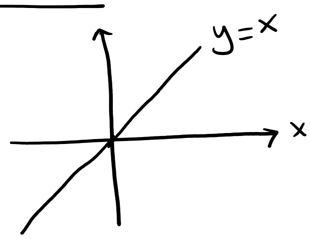
Question 9

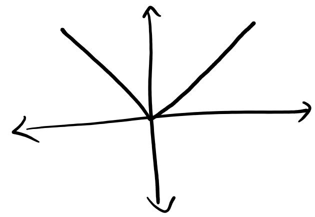
13 Ekim 2020 Salı 16:45

9. (a) How is the graph of $y = f(|x|)$ related to the graph of f?
   (b) Sketch the graph of $y = \sin|x|$.
   (c) Sketch the graph of $y = \sqrt{|x|}$.

Solution:

a)

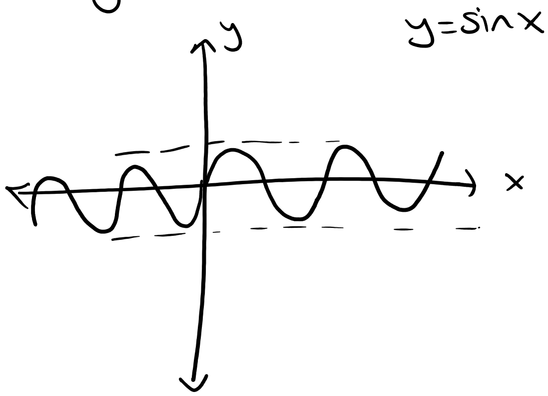




$y = |x|$

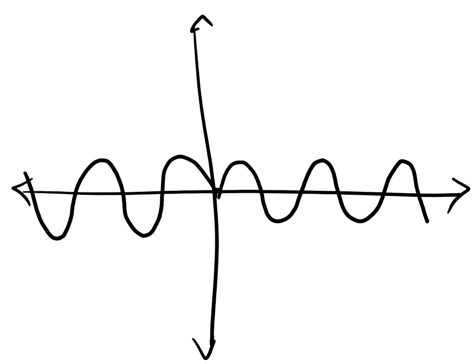
$f(|x|)$ reflects the right side of the graph on y-axis.
(ignore the left part!)

b) $y = \sin|x|$



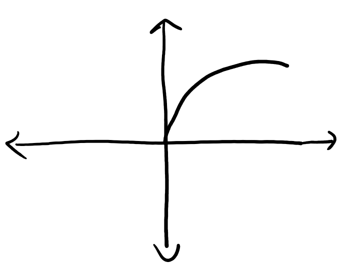
$y = \sin x$

$\Longrightarrow$



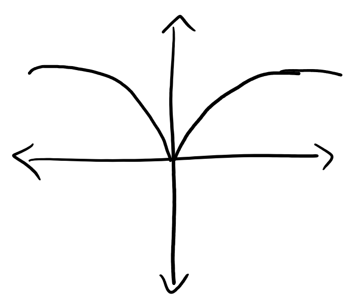
$y = \sin|x|$

c) $y = \sqrt{|x|}$



$y = \sqrt{x}$

$\Longrightarrow$



$y = \sqrt{|x|}$

Question 10

13 Ekim 2020 Salı 16:45

10. (a) Find $f \circ g$ and **its domain** where $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{x-1}{x+3}$.

(b) Given $F(x) = \sin^2(x-5)$, find functions f, g and h such that $F = f \circ g \circ h$.

Solution :

a) $f\circ g(x) = f(g(x)) = f\left(\dfrac{x-1}{x+3}\right) = \dfrac{x-1}{x+3} + \dfrac{1}{\frac{x-1}{x+3}} = \dfrac{(x-1)^2 + (x+3)^2}{(x-1)(x+3)}$ (Note that the func is undfn. at $x=1$, $x=-3$)

$D_{f\circ g} = \{ x \in \mathbb{R} : x \neq -3 \text{ and } x \neq 1 \}$

b) $F(x) = \sin^2(x-5)$ $\quad F = f\circ g\circ h = f(g(h(x)))$

$= f(g(x-5))$

$= f(\sin(x-5))$

$= \sin^2(x-5)$

$\left\{ \begin{array}{l} * f(x) = x^2 \\ * g(x) = \sin x \\ * h(x) = x-5 \end{array} \right.$

Choose the funcs. like that