Recitation 01: Preliminaries

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Math 119 - Calculus with Analytic Geometry

Course webpage: http://ma119.math.metu.edu.tr/

METU | Mathematics Department

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Topics to be covered: (Oct. 12-16)

0.1 Real Numbers and the Real Line
0.2 Cartesian Coordinates in the Plane
0.3 Graphs of Quadratic Equations
0.4 Functions and Their Graphs
0.5 Combining Functions to Make New functions
0.6 Polynomials and Rational Functions
0.7 The Trigonometric Functions

Question 1

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1. Solve the following inequalities.

$$(a)\ \frac{1}{2x+1} \geq 1 - x, \quad (b)\ |x+3| - 2 > 3x, \quad (c)\ \frac{x^3 - x^2 + 4}{x+3} \leq 1$$

Solution:

a) $\frac{1}{2x+1} \geq 1-x \Rightarrow \frac{1}{2x+1} + x - 1 \geq 0 \Rightarrow \frac{\overbrace{1+(2x+1)(x-1)}}{2x+1} \geq 0$

$\frac{2x^2 - x}{2x+1} \geq 0 \Rightarrow$ Let's make a sign table for the factors of the fraction

Note that fraction is undefined at $x = -1/2$

	$-1/2$	0	$1/2$	
x	$-$	$-$	$+$	$+$
$2x-1$	$-$	$-$	$-$	$+$
$2x+1$	$-$	$+$	$+$	$+$
fraction	$-$	$+$	$-$	$+$

Solution set is in the interval:

$(-\frac{1}{2}, 0] \cup [\frac{1}{2}, \infty)$

$\cup$: union

b) $|\overbrace{x+3}| - 2 > 3x$

(*) Note:

$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

i) if $x > -3 \Rightarrow x+3-2 > 3x$
$(x+3>0)$
$\Rightarrow 2x < 1$
$\Rightarrow x < 1/2 \Rightarrow (-3, \frac{1}{2})$

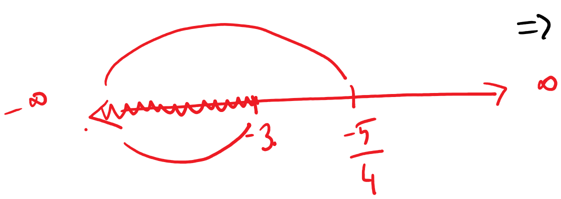
OR

ii) if $x \leq -3 \Rightarrow -(x+3) - 2 \geq 3x$
$x+3 \leq 0$
$\Rightarrow 4x \leq -5$
$\Rightarrow x \leq \frac{-5}{4} \Rightarrow (-\infty, -3]$

Solution set is in the interval.

$(-\infty, -3] \cup (-3, \frac{1}{2})$

$= (-\infty, \frac{1}{2})$



c) $x^3 - x^2 + 4 \quad \leq 1 \Rightarrow x^3 - x^2 + 4 \quad \leq 0$

c) $\dfrac{x^3-x^2+4}{x+3} \leq 1 \Rightarrow \dfrac{x^3-x^2+4}{x+3} - 1 \leq 0$

$\Rightarrow \dfrac{x^3-x^2-x+1}{x+3} \leq 0 \Rightarrow \dfrac{x^2(x-1)-(x-1)}{x+3} \leq 0 \Rightarrow \dfrac{\overbrace{(x-1)(x^2-1)}^{(x-1)^2\cdot(x+1)\ =\ (x-1)(x+1)}}{x+3} \leq 0$

To find the sign of the fraction, let's make a sign table.

	-3	-1	1	
$(x-1)^2$	+	+	+	+
$x+1$	−	−	+	+
$x+3$	−	+	+	+
fraction	+	−	+	+

Note that fraction is undfn at $x=-3$

Solution set is in the interval

$(-3,-1] \cup \{1\}$

Question 2

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x_0 y_0 x_1 y_1

2. Write an equation for the line through the points $(-1, 5)$ and $(0,3)$.

Solution:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{3-5}{0-(-1)} = -2$$

Note: point-slope line eqn.

$$\boxed{y - y_0 = m.(x - x_0)}$$

$(x_0, y_0) \to$ point

$m \to$ slope

$$y - y_0 = m.(x - x_0)$$

$$y - 5 = -2.(x+1) \Rightarrow \boxed{y = -2x + 3}$$

Question 3

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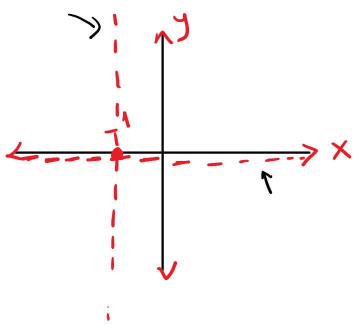
3. Find the equation for the (a) vertical line and (b) the horizontal line through the point $(-1, 0)$.

Solution:



a) $x=-1$ is the vertical line.

b) $y=0$ is the horizontal line.

Question 4

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4. Find the equation for the line through $P(-1,3)$ that is perpendicular to the line $y+x+2=0$. Find the x and y-intercepts of this line.

L2

Solution:

$L_1: y=-x-2 \Rightarrow m_1=-1$

we know that $L_1 \perp L_2 \Rightarrow \overset{-1}{m_1} \cdot m_2 = -1$

$\boxed{m_2 = 1}$ $\boxed{P(-1,3)}$

$y-y_0 = m.(x-x_0) \Rightarrow y-3 = 1.(x+1)$

$\boxed{y = x+4}$

Note: L_1 and L_2 : lines
m_1 and m_2 : slopes

1. $L_1 \perp L_2 \Rightarrow m_1 \cdot m_2 = -1$
2. $L_1 // L_2 \Rightarrow m_1 = m_2$

Question 5

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5. Describe and sketch the regions defined by the followings

(a) $x^2 + y^2 - 2x - 4y \le 4$, (b) $x^2 + y^2 \le 4$, $x^2 + y^2 > 2y$, (c) $x^2 + y^2 > 2y$, $y > 1 + x$

(d) $y > (x-1)^2 + 2$, $y < 2x$, (e) $4x^2 + (y-2)^2 \le 4$

Solution:

a) $x^2+y^2-2x-4y \le 4$

$x^2-2x+1+y^2-4y+4 \le 9$

$(x-1)^2+(y-2)^2 \le 9$

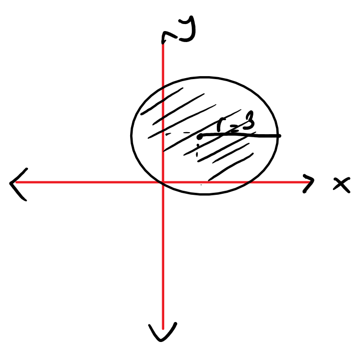
the circle with centre $(1,2)$ radius $r=3$

and its interior

* Note:

$(x-a)^2+(y-b)^2=r^2$

is defined the circle with centre (a,b) radius r



b) $x^2+y^2 \le 4$, $x^2+y^2 > 2y$

$\Downarrow$

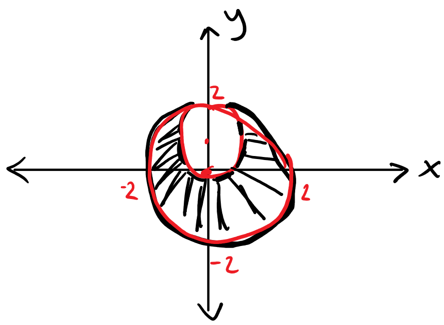
the disk with centre $(0,0)$ $r=2$

$x^2+y^2-2y+1 > 1$

$x^2+(y-1)^2 > 1$

the outside of the circle centre $(0,1)$

centre $(0,1)$
$r=1$



c) $x^2+y^2>2y$, $y>1+x$

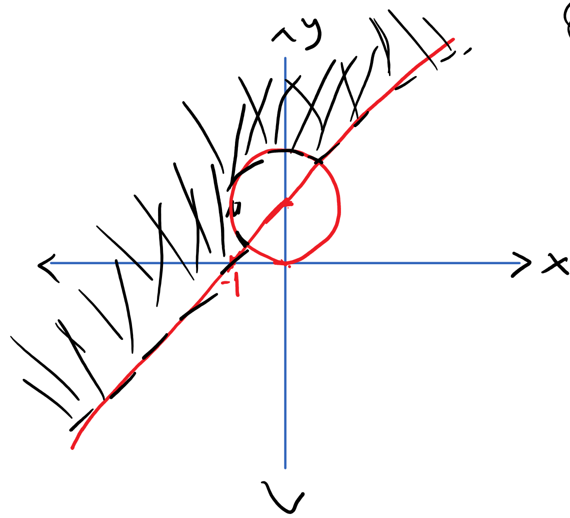
$0<1+0$
$\Downarrow$

$x^2+y^2-2y+1>1$

$x^2+(y-1)^2>1$

the circle with centre $(0,1)$
$r=1$
(outside of circle without boundries)

$(y=1+x)$
$\Rightarrow$ $(0,1)$
$(-1,0)$
it is a line.



(d) and (e) is for you.

Question 6

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6. Find the points of intersection of the pairs of curves

(a) $y = x^2 + 3$, $y = 3x + 1$ (b) $2x^2 + 2y^2 = 5$, $xy = 1$

Solution

a) To find the intersection points, we need solve together:

$$\left.\begin{array}{l} y = x^2+3 \\ y = 3x+1 \end{array}\right\} \Rightarrow x^2+3 = 3x+1 \Rightarrow x^2-3x+2=0 \Rightarrow (x-2)(x-1)=0$$

$$\begin{array}{ll} x & -2 \\ x & -1 \end{array}$$

$$\Rightarrow \left.\begin{array}{ll} x=2 & \text{and} \quad x=1 \\ \Downarrow & \qquad \Downarrow \\ y=7 & \qquad y=4 \end{array}\right\}$$ $(2,7)$ and $(1,4)$ are the intersection points.

b) $2x^2+2y^2=5$, $xy=1$

To find intersection points, we need to solve ineq. together.
Observe that $\boxed{y \neq 0}$ and $\boxed{x \neq 0}$. (Since this points don't satify ineq.)

$xy = 1 \Rightarrow y = \frac{1}{x}$

$$2x^2 + 2\cdot\frac{1}{x^2} = 5 \Rightarrow \frac{2x^4+2}{x^2} - 5 = 0 \Rightarrow \frac{2x^4-5x^2+2}{x^2} = 0$$

$$\Rightarrow 2x^4 - 5x^2 + 2 = 0 \Rightarrow (2x^2-1)(x^2-2) = 0 \Rightarrow x_{1,2} = \mp\frac{1}{\sqrt{2}} \Rightarrow y_{1,2} = \mp\sqrt{2}$$

$$\begin{array}{ll} 2x^2 & -1 \\ x^2 & -2 \end{array}$$

$$x_{3,4} = \mp\sqrt{2} \Rightarrow y_{3,4} = \mp\frac{1}{\sqrt{2}}$$

intersection points:

$\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ $\left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$ $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$ $\left(-\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$

Question 7

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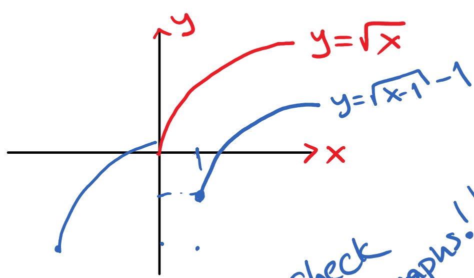
• $y=|x|$

• $y=\frac{1}{x}$ • $y=\sin x$ • $y=\cos x$ • $y=x^2$

7. Write an equation of the graph obtained by shifting the graph of $y=\sqrt{x}$

(a) down 1, right 1 (b) down 2, left 4 (c) up 2, left 1 (d) up 1, right 1



Check the graphs!!

a) $y=\sqrt{x-1}-1$

b) $y=\sqrt{x+4}-2$

c) $y=\sqrt{x+1}+2$

d) $y=\sqrt{x-1}+1$

Note:

$\rightarrow 1$ $g(x-1)$

$\leftarrow 1$ $g(x+1)$

$\uparrow 1$ $g(x)+1$

$\downarrow 1$ $g(x)-1$

Question 8

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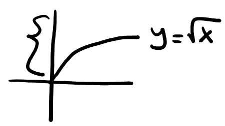
8. Find the domain and range of each function and sketch their graphs:

(a) $f(x) = \sqrt{x^2 - 1}$ (b) $f(x) = \dfrac{1}{|2-x|}$ (c) $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$

Solution:



a) $f(x) = \sqrt{x^2-1}$

$x^2 - 1 \geqslant 0 \Rightarrow x^2 \geq 1 \Rightarrow |x| \geq 1 \Rightarrow x \geq 1$ OR $x \leq -1$

$D_f = \{x \in \mathbb{R} : x \geq 1 \text{ OR } x \leq -1\}$

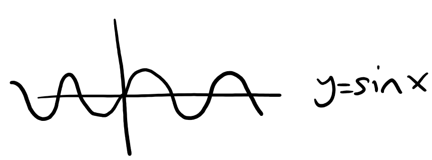
$R_f = \{y \in \mathbb{R} : y \geq 0\}$

b) $f(x) = \dfrac{1}{|2-x|}$

$D_f = \{x \in \mathbb{R} : x \neq 2\}$ Since fraction is undfn. at $x=2$

$R_f = \{y \in \mathbb{R} : y > 0\}$

c) $f(x) = 1 + \sin\left(x + \frac{\pi}{4}\right)$



$D_f = \mathbb{R}$

$R_f = [0, 2]$

$-1 \leq \sin(?) \leq 1$

$0 \leq 1 + \sin\left(x + \frac{\pi}{4}\right) \leq 2$

Question 9

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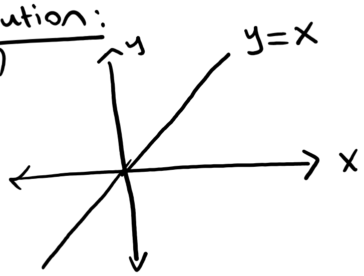
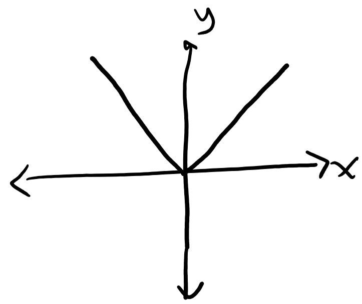
9. (a) How is the graph of $y = f(|x|)$ related to the graph of f?
   (b) Sketch the graph of $y = \sin|x|$.
   (c) Sketch the graph of $y = \sqrt{|x|}$.

Solution:

a)

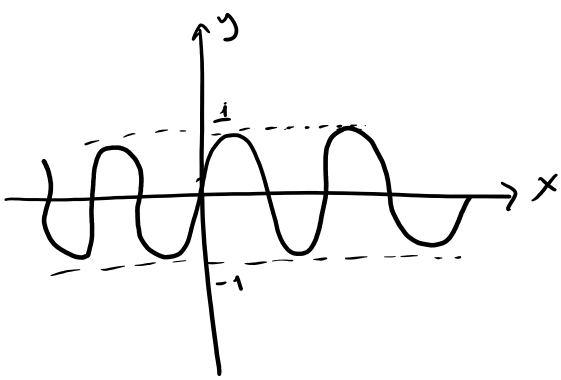


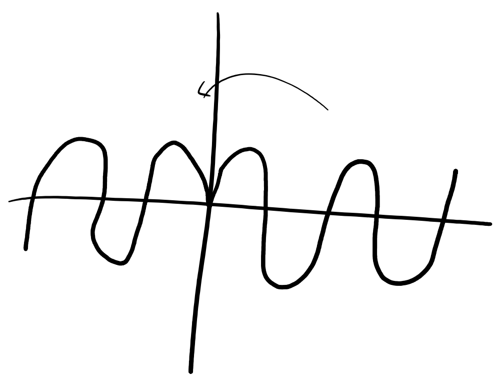


$y=|x|$

$f(|x|)$ reflects the rightside of graph to the left side. on y-axis.

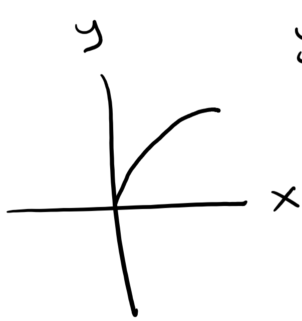
b) $y=\sin|x|$



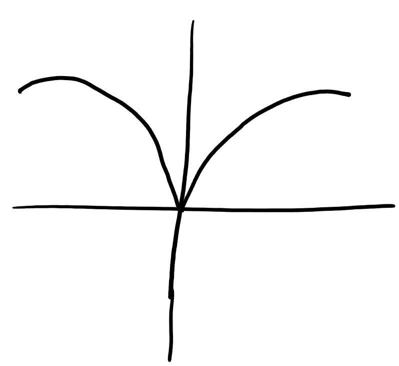


$y=\sin|x|$

c) $y=\sqrt{|x|}$



$y=\sqrt{x}$



$y=\sqrt{|x|}$

Question 10

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10. (a) Find $f \circ g$ and **its domain** where $f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x-1}{x+3}$.

(b) Given $F(x) = \sin^2(x-5)$, find functions f, g and h such that $F = f \circ g \circ h$.

Solution:

a) $f \circ g(x) = f(g(x)) = f\left(\frac{x-1}{x+3}\right) = \frac{x-1}{x+3} + \frac{1}{\frac{x-1}{x+3}} = \frac{\overbrace{(x-1)^2+(x+3)^2}^{2x^2+x+4}}{(x-1)(x+3)}$ is undfn. at $x=1$, $x=-3$

$D_{f \circ g} = \{x \in \mathbb{R} : x \neq 1 \text{ and } x \neq -3\}$

b) $F = f(\overbrace{g(\overbrace{h(x)}^{x-5})}^{\sin(x-5)}) = \sin^2(x-5)$

✓ $h(x) = x-5$

✓ $g(x) = \sin(x)$

✓ $f(x) = x^2$