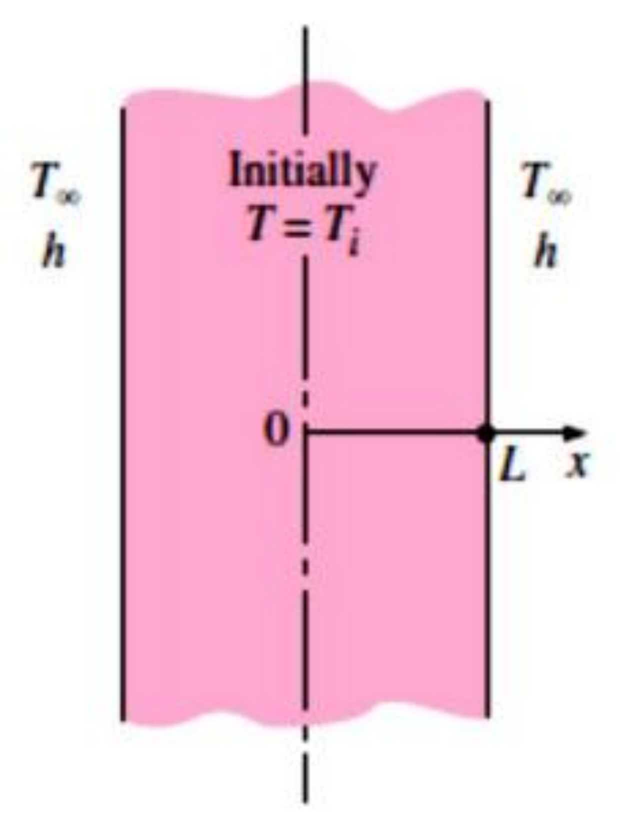
5. Transient Heat Conduction



One-dimensional, transient, differential equation of heat conduction for a slab in Cartesian coordinates with constant k and no heat generation is:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \quad , \quad \alpha = \frac{k}{\rho \, c_p}$$

The exact (analytical) solution is beyond our scope. Numerical solutions for simple cases will be given later.

5.1 Transient Temperature Charts – Heisler Charts

The temperature distribution for one-dimensional transient heat conduction in simple geometries such as a slab, cylinder and sphere have been calculated and are available in the form of transient temperature charts, called **Heisler Charts**.

The Heisler Charts are a collection of two charts per contained geometry developed in 1947 by M. P. Heisler and expanded in 1961 by H. Gröber with a triple chart per geometry, a plane wall (slab), cylinder, and sphere. The temperature distribution is plotted as a function of time and position.

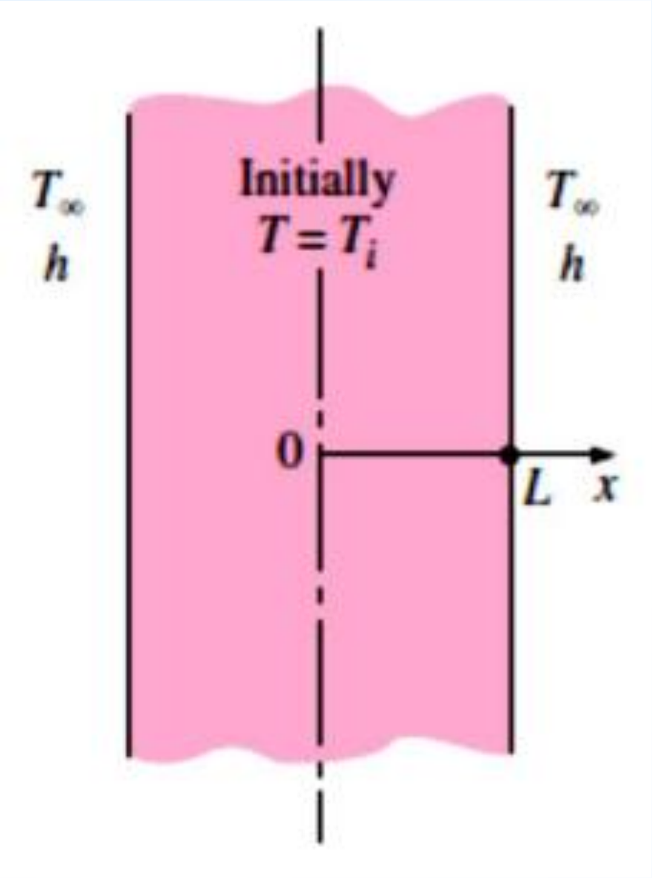
The three charts associated with each geometry, slab, cylinder, and sphere, are:

* The temperature, T_0, at the center of the geometry at a given time t.
* The temperature at other locations at the same time in terms of T_0.
* The total amount of heat transfer up to the time t.

The Heisler Charts can only be used when:

- The body is initially at a *uniform temperature;*
- The temperature of the medium surrounding the body is *constant and uniform;*
- The convection heat transfer coefficient is *constant and uniform; and*
- *There is no heat generation in the body.*

5.1.1 Heisler Charts for a Slab with Constant k



Differential Equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \quad \text{in } -L \le x \le L \text{ and } t > 0$$

IC: $T(x,0) = T_i$ in $-L \le x \le L$ and $t = 0$

BC's:

(1) $\left.\frac{\partial T}{\partial x}\right|_{x=0,\, t>0} = 0$

(2) $-k \left.\frac{\partial T}{\partial x}\right|_{x=L,\, t>0} = h\left(T(L,t) - T_\infty\right)$

In order to use the charts, we need to define a new non-dimensional temperature $\theta(x,t)$ as

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

Differential Equation: $$\frac{\partial^2 \theta(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta(x,t)}{\partial t} \quad \text{in } -L \le x \le L \text{ and } t > 0$$

Initial Condition: $$\theta = 1 \quad \text{in } -L \le x \le L \text{ and } t = 0$$

Boundary Conditions:

(1) $$\frac{\partial \theta}{\partial x} = 0 \quad \text{at } x = 0 \,,\ t > 0$$

(2) $$k \frac{\partial \theta}{\partial x} + h\,\theta = 0 \quad \text{at } x = L \,,\ t > 0$$

Define non-dimensional variables: $X = \dfrac{x}{L}$ and $\theta(x,t) = \dfrac{T(x,t) - T_\infty}{T_i - T_\infty}$

Define non-dimensional parameters:

$\tau = \dfrac{\alpha\, t}{L^2} = Fo$ Fourier number

$\dfrac{h\,L}{k} = Bi$ Biot number

Differential Equation: $$\frac{\partial^2 \theta(X,\tau)}{\partial X^2} = \frac{\partial\, \theta(X,\tau)}{\partial \tau} \quad \text{in } 0 \le X \le 1 \text{ and } \tau > 0$$

Initial Condition: $\theta(X,0) = 1$ in $0 \le X \le 1$ and $\tau = 0$

Boundary Conditions: $$\left.\frac{\partial \theta}{\partial X}\right|_{X=0,\ \tau>0} = 0 \quad \text{and} \quad \left.\frac{\partial \theta}{\partial X}\right|_{X=1,\ \tau>0} = -Bi\ \theta(1,\tau)$$

Remember the definitions of Biot number and Fourier number:

Biot number $$Bi = \frac{h\,L_c}{k} = \frac{\dfrac{L_c}{k\,A}}{\dfrac{1}{h\,A}} = \frac{R_{cond}}{R_{conv}}$$

Characteristic Length $$L_c = \frac{\text{Volume, } V}{\text{Heat transfer surface area, } A}$$

Fourier number $$\tau = Fo = \frac{\alpha\,t}{L_c^2} = \frac{\text{Rate of conduction}}{\text{Rate of storage}}$$

Exact Solution (for the slab – using a method called separation of variables)

$$\theta(X,\tau) = \left[A\cos(\lambda X) + B\sin(\lambda X)\right] e^{-\lambda^2 \tau} \qquad \text{in } 0 \le X \le 1 \text{ and } \tau > 0$$

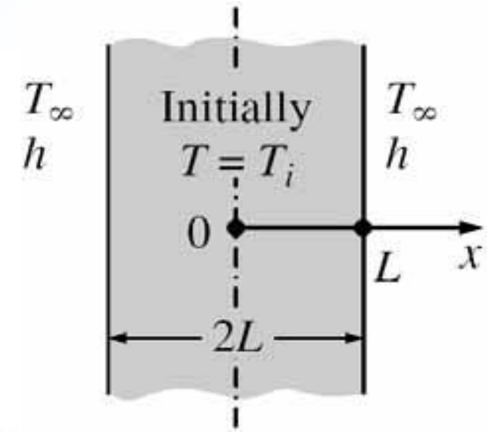
Apply Boundary Conditions and Initial Condition

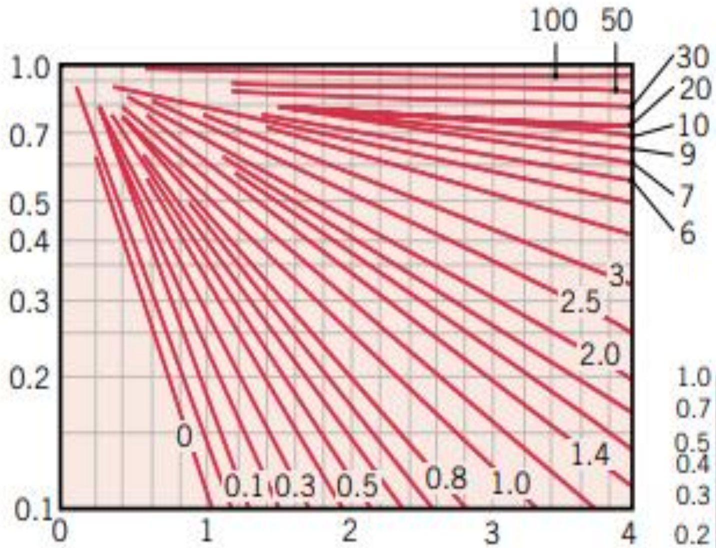
$$\theta(X,\tau) = \sum_{n=1}^{\infty} \left[\frac{4\sin(\lambda_n)}{2\lambda_n + \sin(2\lambda_n)} \cos(\lambda_n X) \right] e^{-\lambda_n^2 \tau} \qquad \text{in } 0 \le X \le 1 \text{ and } \tau > 0$$

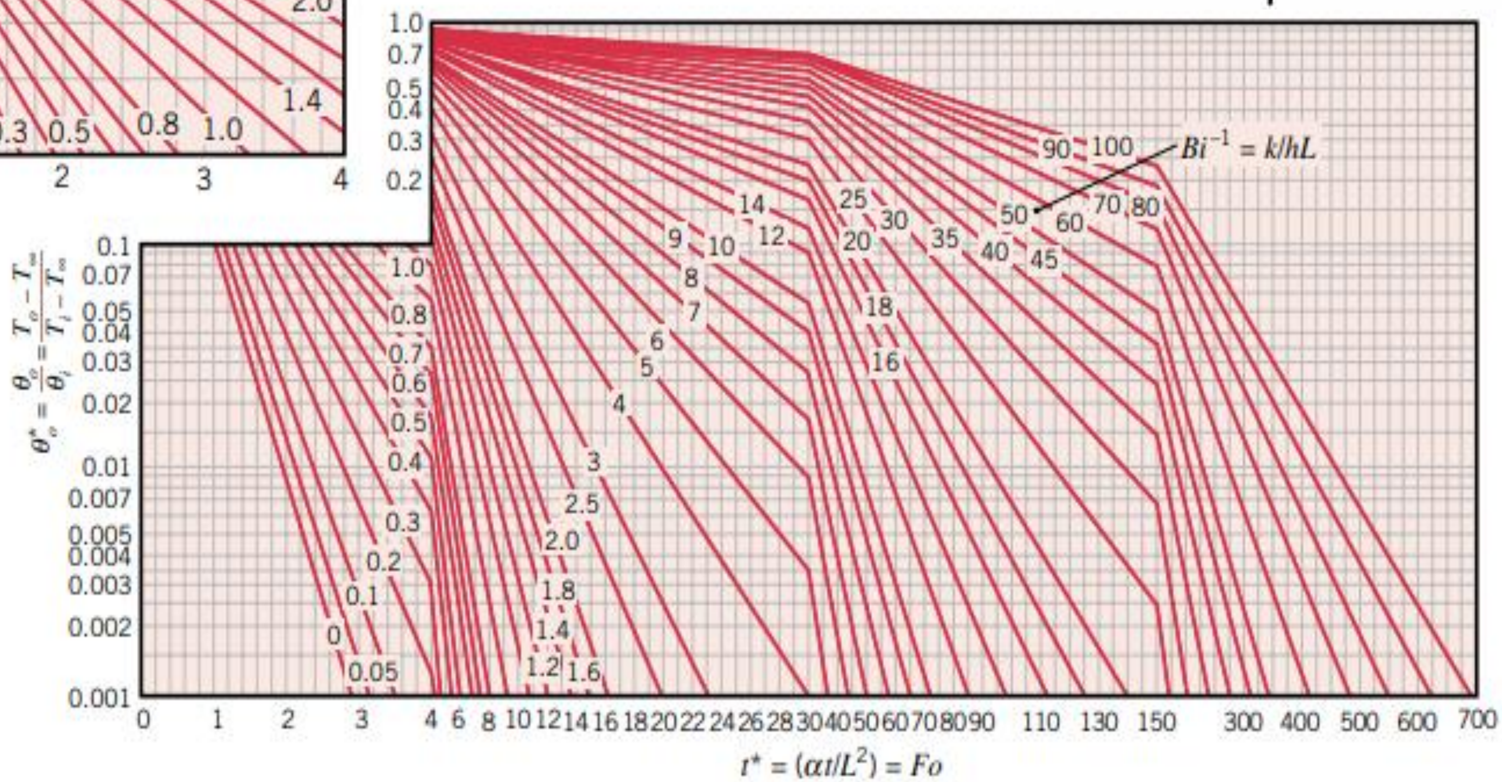
λ_n's (eigenvalues) are the roots of the equation (characteristic equation or eigenfunction) $\quad \lambda_n \tan(\lambda_n) = Bi$

One-term approximation for Fo > 0.2 $\quad \theta(X,\tau) = A_1 \cos(\lambda_1 X)\, e^{-\lambda_1^2 \tau}$

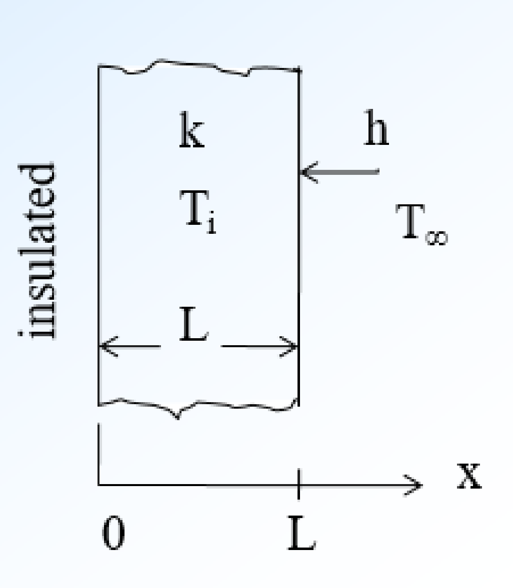
The solution is given in the chart as θ_0 vs $Fo = \alpha t/L^2$ with 1/Bi as the parameter.







Example 1



L = 30 cm

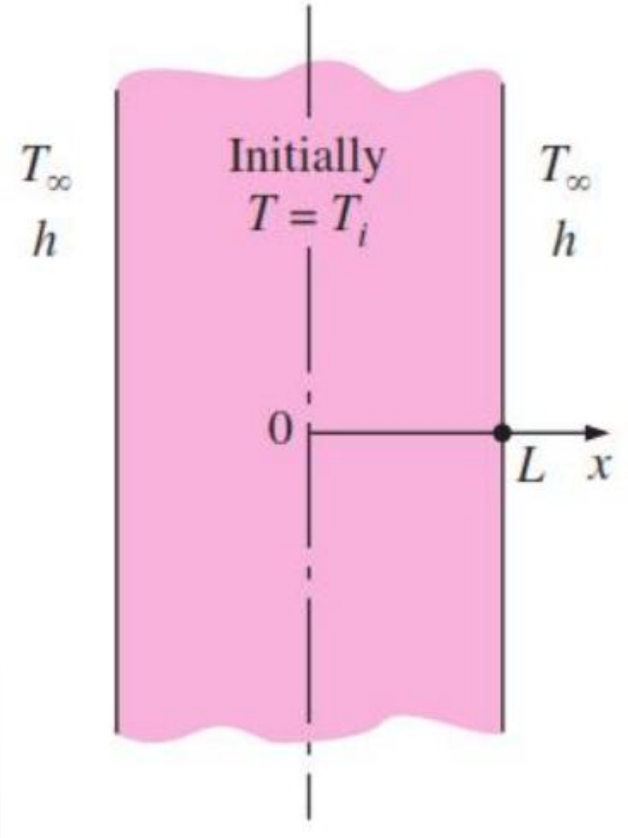
k = 0.865 W/m.K

T_i = 500 °C

α = 1.3 10^{-6} m^2/s

T_∞ = 50 °C

h = 28.4 $W/m^2.K$



Both Figures represent the same problem due to symmetry.

Find the temperature at the insulated surface at x = 0 and time t = 20 hours.

$$\frac{\alpha\, t}{L^2} = \frac{(1.3\ 10^{-6})\,(20)\,(3600)}{(0.3)^2} \cong 1.0$$

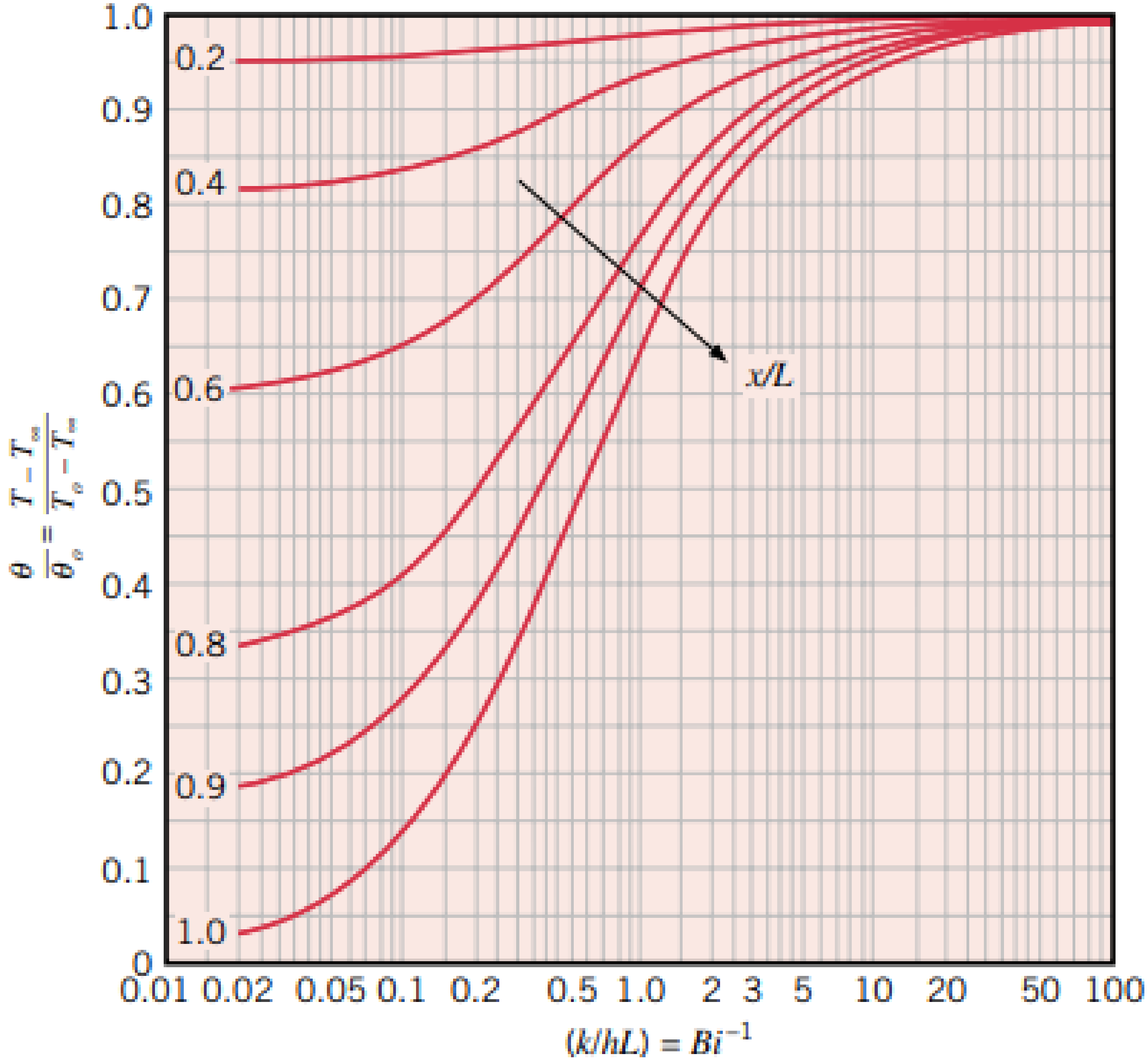
$$\frac{k}{h\, L} = \frac{0.865}{(28.385)\,(0.3)} \cong 0.1$$

Use the first chart

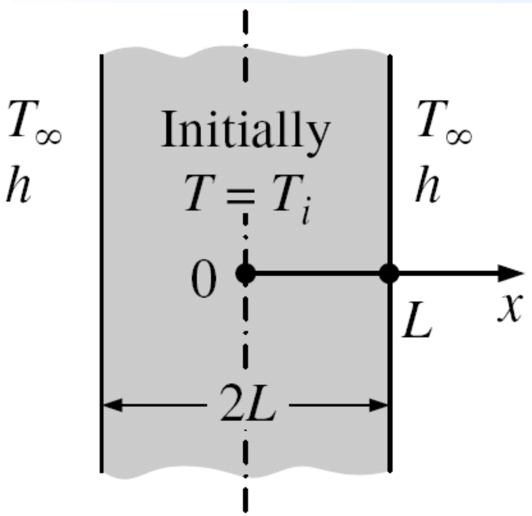
$$\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.18$$

$$T_0 = (0.18)\left(T_i - T_\infty\right) + T_\infty = (0.18)\left(500 - 50\right) + 50$$

$$= 131\ ^\circ C$$



The temperature at a position x/L is found using the second chart given here.



Heat Transfer:

The *maximum amount of heat that a body can gain (or* lose if $T_i = T_\infty$*) occurs when the temperature of the body* changes from the initial temperature T_i *to the* ambient temperature

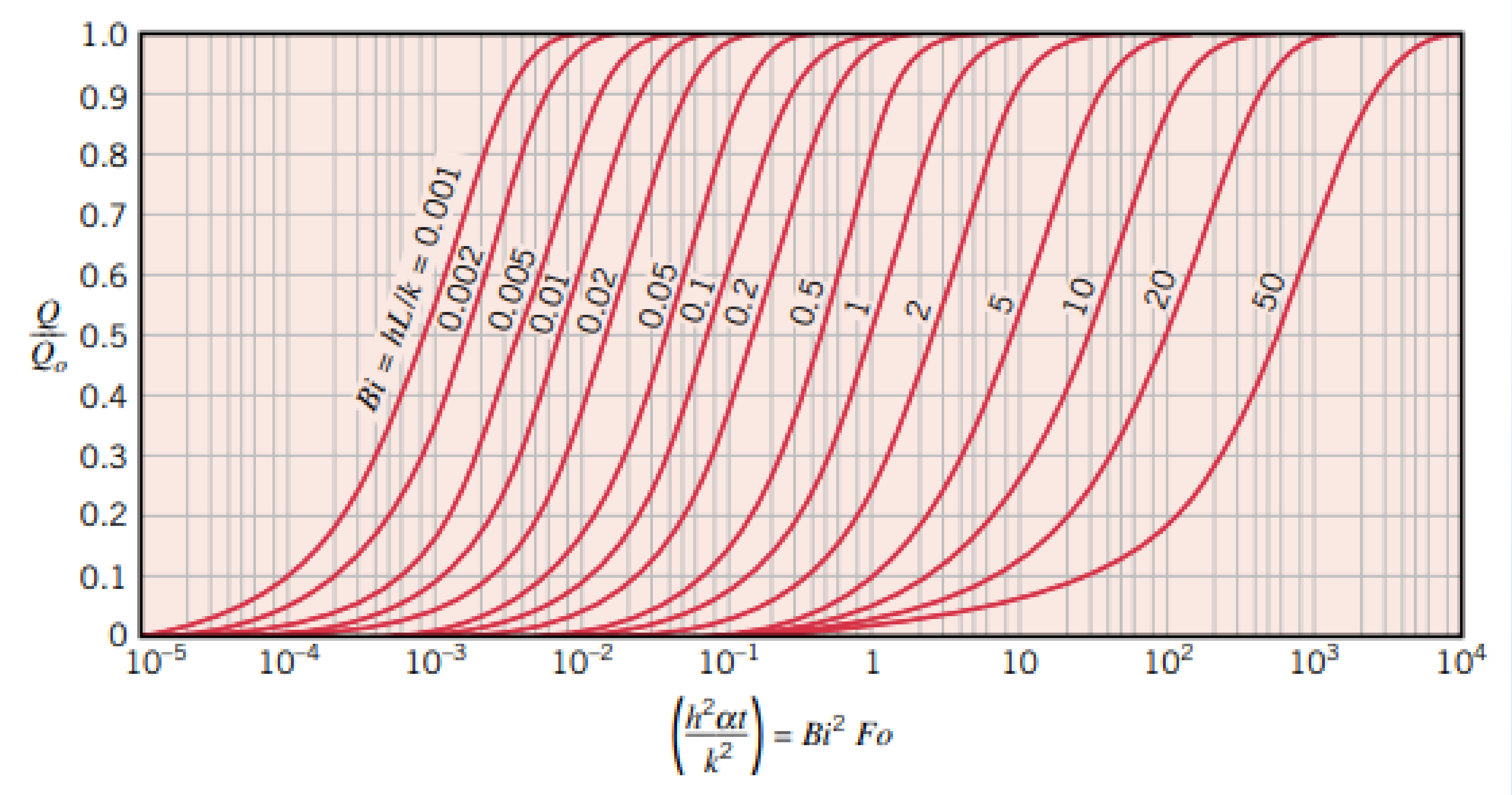
$$Q_{max} = m\, c_p\, (T_i - T_\infty) = \rho\, V\, c_p\, (T_i - T_\infty) \quad \text{in Joules}$$

The amount of heat (thermal energy) transfer, *E , at a finite time t is can* be expressed as

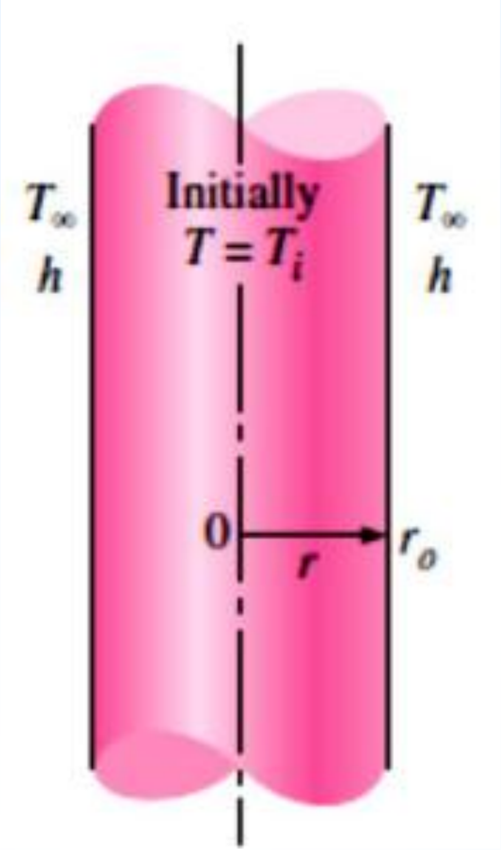
$$Q = \int_0^t \dot{Q}\, dt = \int_0^t \left(-k\, A\, \frac{\partial T}{\partial x} \right) dt$$

The third Heisler chart gives Q / Q_{max}. It is given as Q / Q_0 in the third chart.

Graphical results for the energy transferred from a plane wall over the time interval t are presented in third chart given below.



5.1.2 Heisler Charts for an Infinite Cylinder with Constant k



Differential Equation:

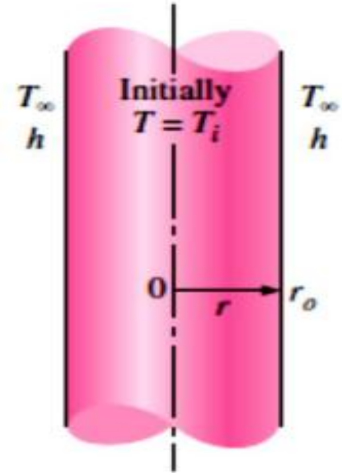
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{in } 0 \le r \le r_0 \text{ and } t > 0$$

IC: $\theta = 1$ in $0 \le r \le r_0$ and $t = 0$

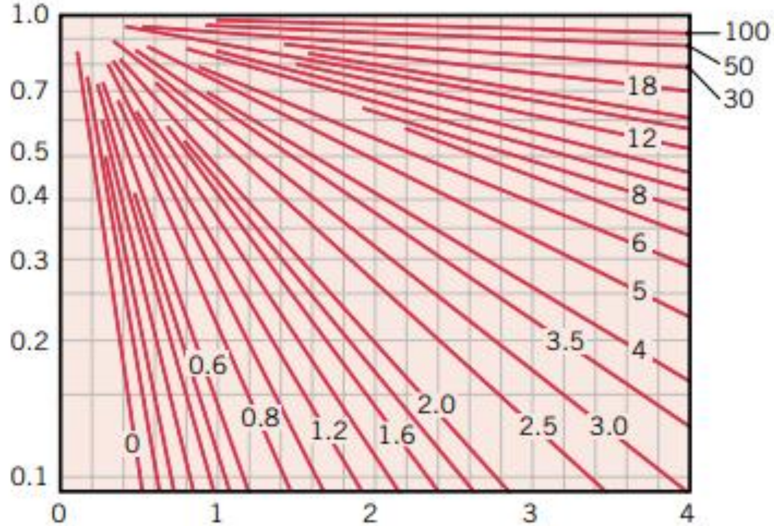
BC's:

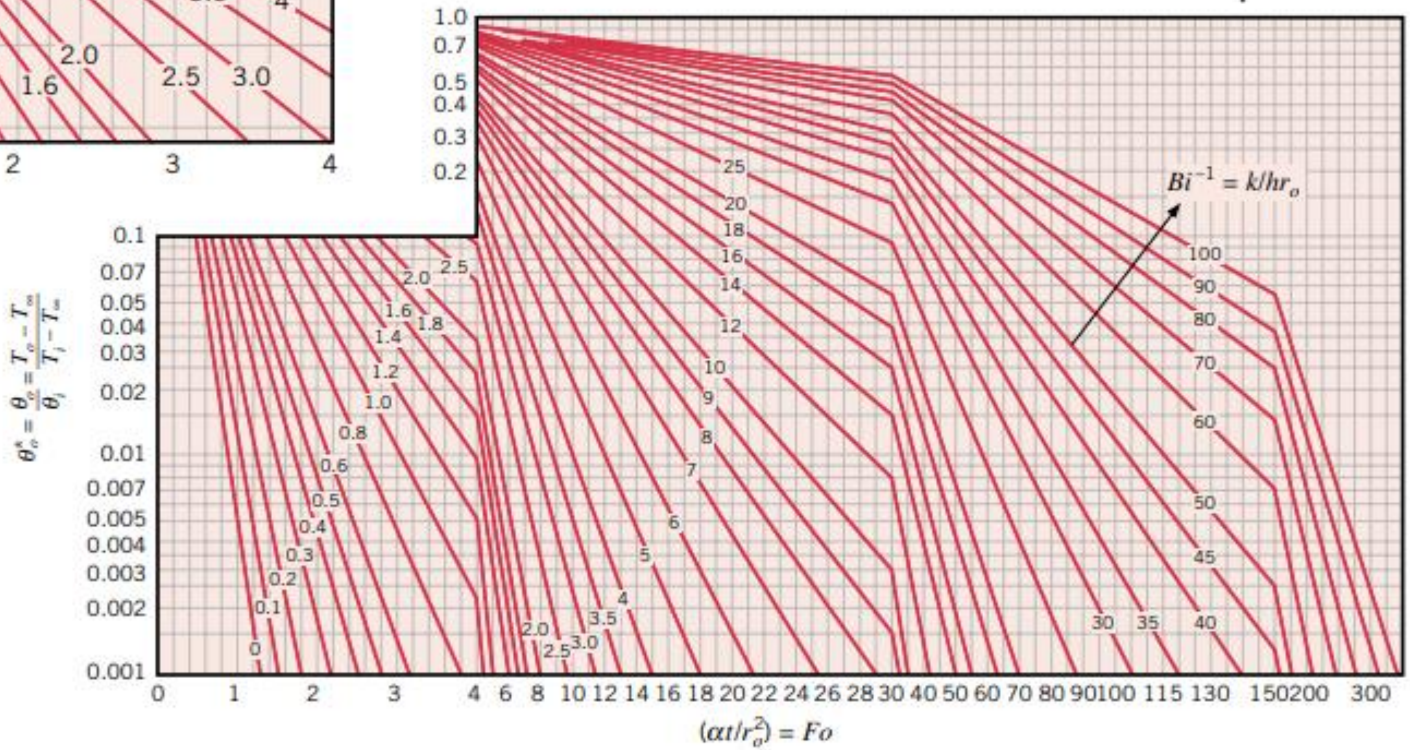
(1) $\frac{\partial \theta}{\partial r} = 0$ at $r = 0$, $t > 0$

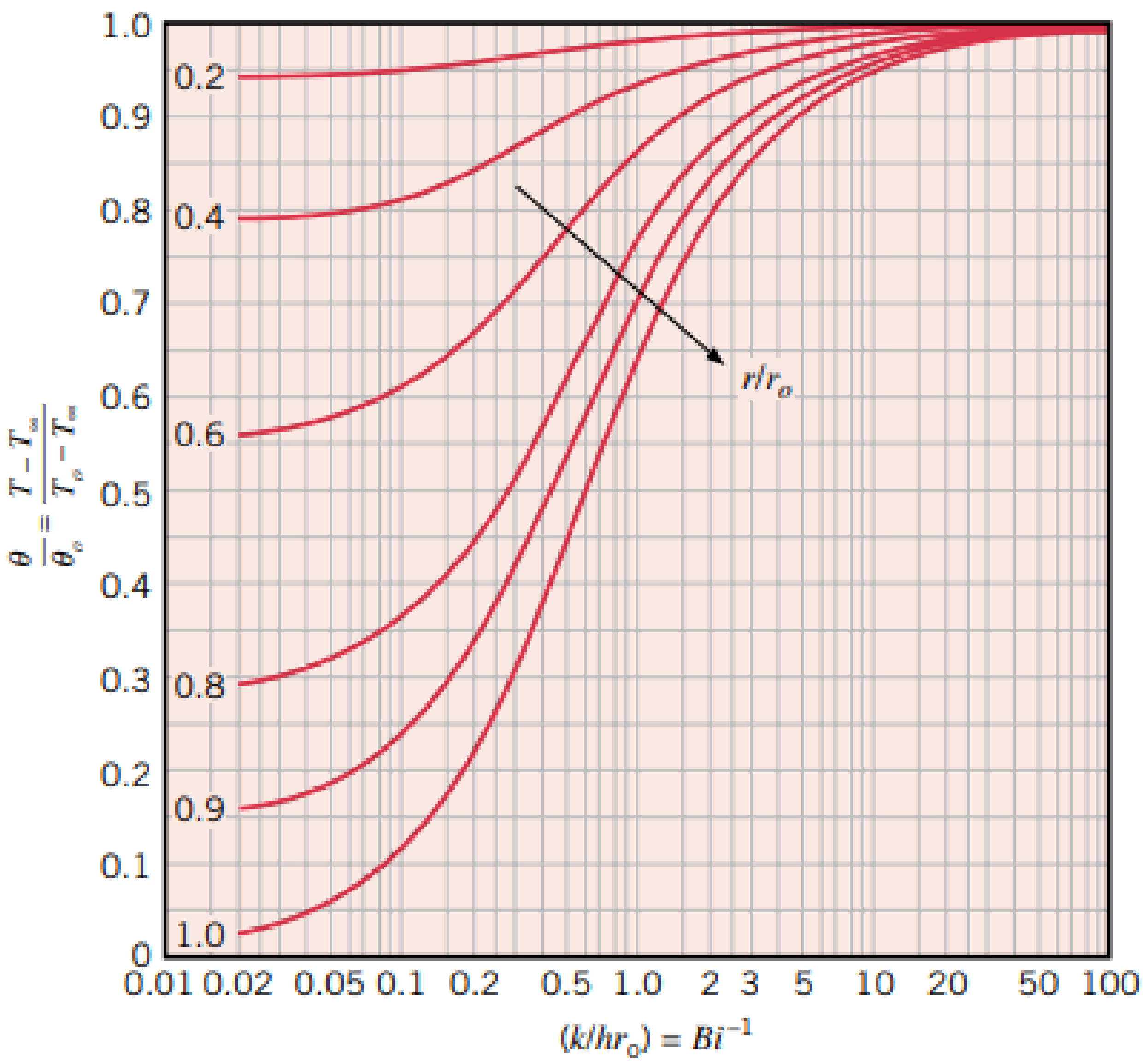
(2) $k\frac{\partial \theta}{\partial r} + h = 0$ at $r = r_0$, $t > 0$



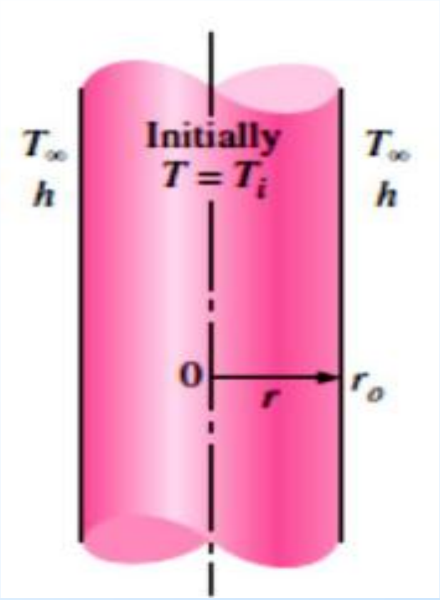
The solution is given in the chart as θ_0 vs $Fo = \alpha\, t/L^2$ with 1/Bi as the parameter.



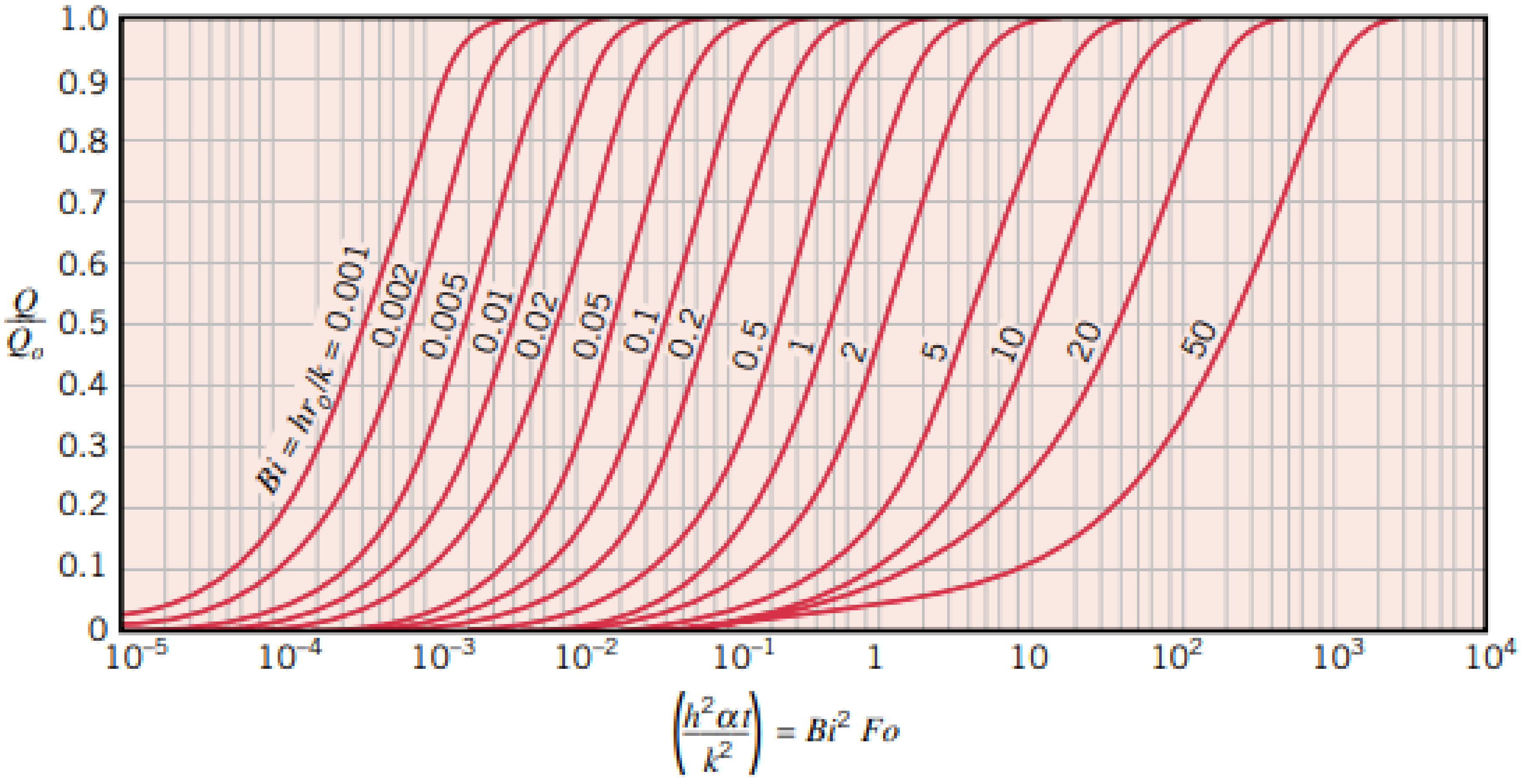




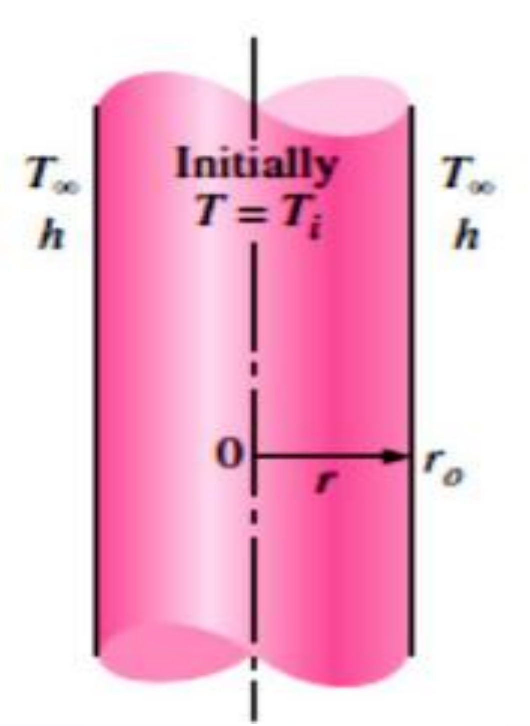
The temperature at a position r/r_0 is found using the second chart given here.



Graphical results for the energy transferred from a infinite cylinder over the time interval t are presented in third chart given below.



Example 2



r_0 = 2.5 cm

k = 215 W/m.K

T_i = 200 °C

α = 18.4 10^{-5} m^2/s

ρ = 2700 kg/m^3

c = 0.9 kJ/kg.K

T_∞ = 50 °C

h = 525 $W/m^2.K$

Calculate

(a) Temperature at r = 1.2 cm

(b) Heat loss per unit pipe length

after 1 minute exposure to the environment

$$\frac{\alpha\ t}{r_0^2} = \frac{(8.4\ 10^{-5})\ (60)}{(0.025)^2} = 8.064$$

$$\frac{k}{h\ r_0} = \frac{215}{(525)\ (0.025)} = 16.38$$

Use the first chart

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.38$$

$$T_0 = (0.38)\ \left(T_i - T_\infty\right) + T_\infty = (0.38)\ \left(200 - 70\right) + 70 = 119.4\ ^\circ C$$

$$\frac{k}{h\ r_0} = 16.38$$

$$\frac{r}{r_0} = \frac{1.2}{2.5} = 0.48$$

Use the scond chart

$$\frac{T - T_\infty}{T_0 - T_\infty} = 0.98$$

$$T = (0.98)\ \left(T_0 - T_\infty\right) + T_\infty = (0.98)\ \left(119.4 - 70\right) + 70 = 118.4\ ^\circ C$$

$$\frac{h^2 \alpha t}{k^2} = \frac{(525)(8.4\ 10^{-5})(60)}{(215)^2} = 0.03$$

$$\frac{h r_0}{k} = \frac{(525)(0.025)}{215} = 0.061$$

Use the third chart

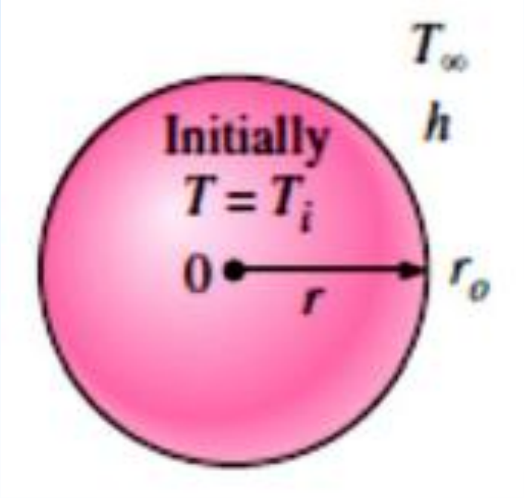
$$\frac{Q}{Q_0} = 0.65$$

$$Q_0 = \rho c V (T_i - T_\infty)$$

$$\frac{Q_0}{L} = \rho c \frac{\pi r_0^2 L}{L} (T_i - T_\infty) = (2700)(900)\pi(0.025)^2 (200 - 70) = 6.203\ 10^5 \text{ J/m}$$

$$\frac{Q}{L} = \frac{Q_0}{L}(0.65) = 4.032 \text{ J/m}$$

5.1.2 Heisler Charts for a sphere with Constant k



Differential Equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \qquad \text{in } 0 \le r \le r_0 \text{ and } t > 0$$

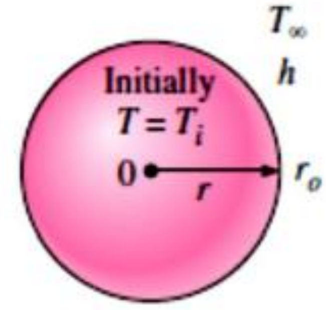
$$\theta = 1 \qquad \text{in } 0 \le r \le r_0 \text{ and } t = 0$$

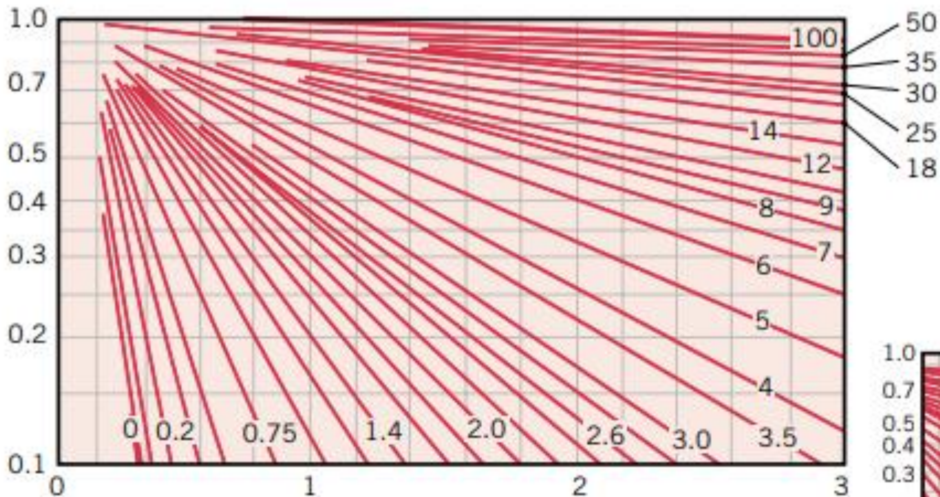
BC's:

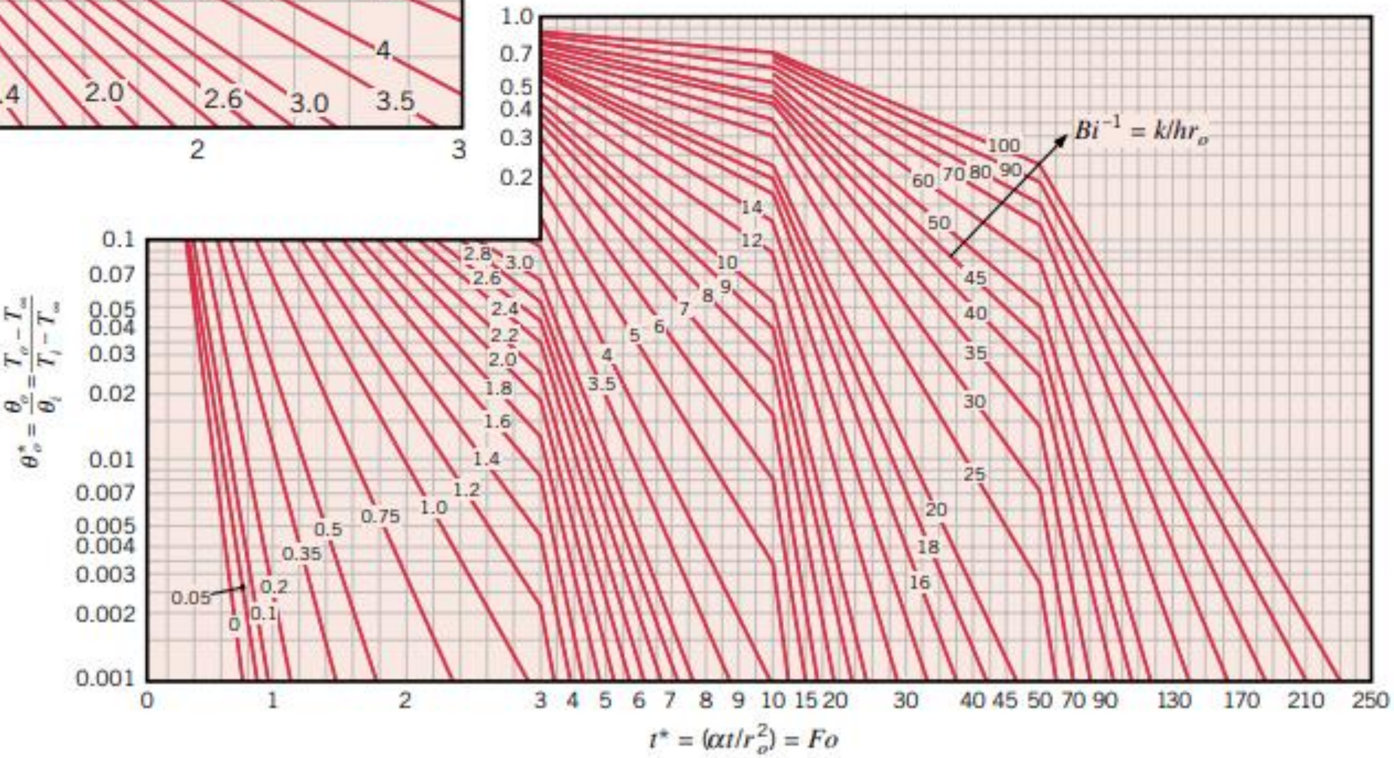
(1) $\dfrac{\partial \theta}{\partial r} = 0$ at $r = 0$ $t > 0$

(2) $k\dfrac{\partial \theta}{\partial r} + h = 0$ at $r = r_0$ $t > 0$

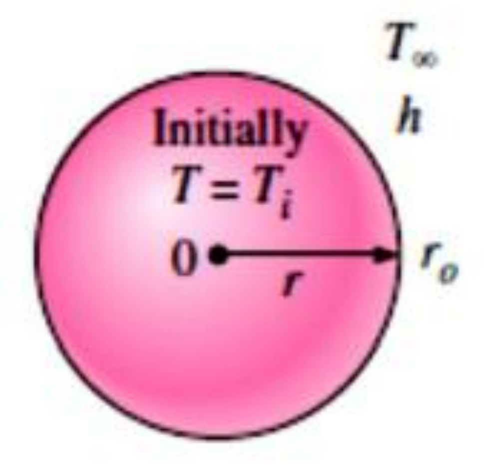
The solution is given in the chart as θ_0 vs $Fo = \alpha t/L^2$ with 1/Bi as the parameter.







Example 3



r_0 = 10 cm

k = 50 W/m.K

T_i = 250 °C

c = 0.9 kJ/kg.K

T_∞ = 10 °C

h = 280 W/m^2.K

How long does it take for the center temperature, T_0, to reach 150 °C?

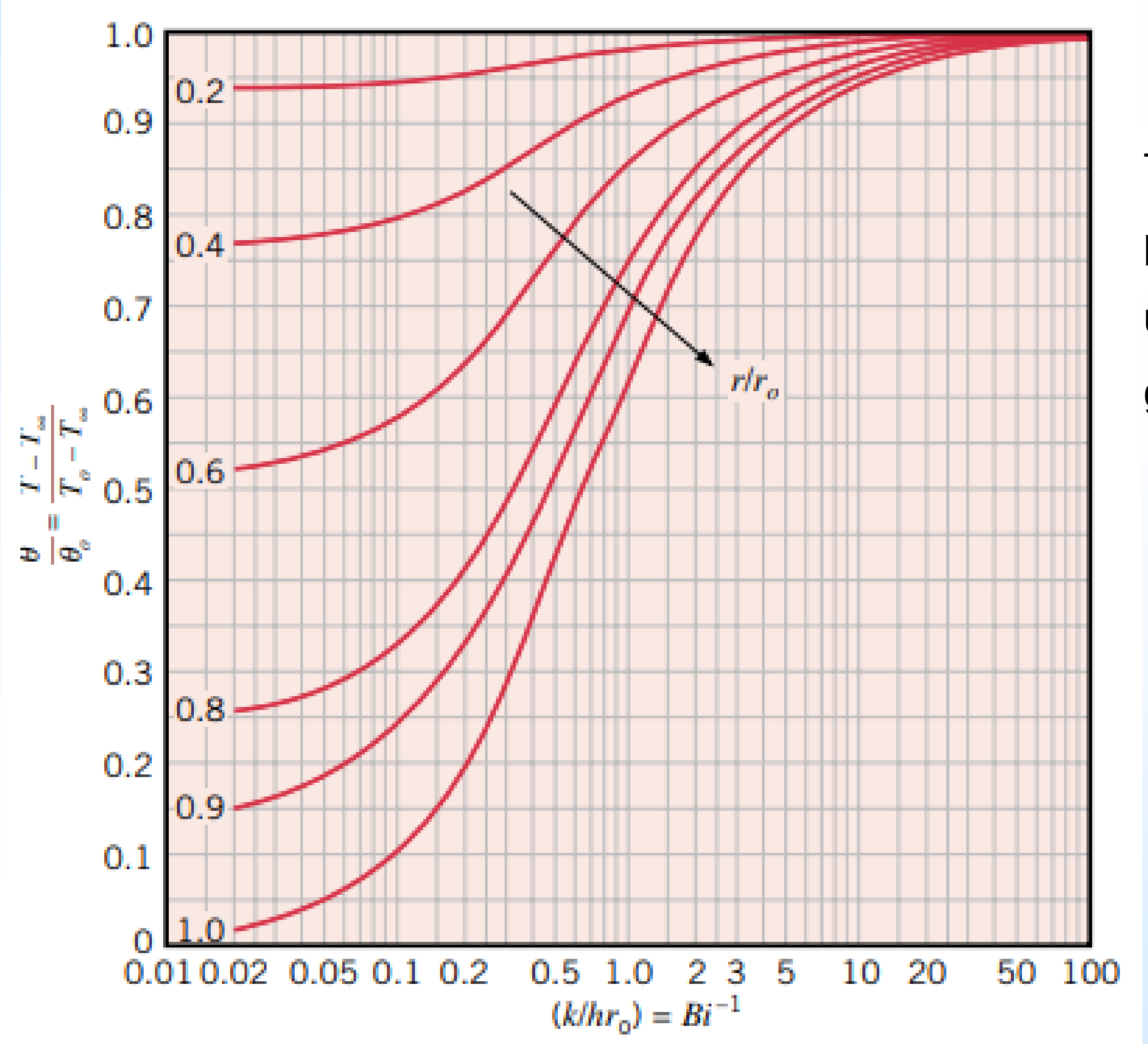
$$\frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{150 - 10}{250 - 10} = 0.5833$$

$$\frac{k}{h\,L} = \frac{0.865}{(28.385)\,(0.3)} \cong 0.1$$

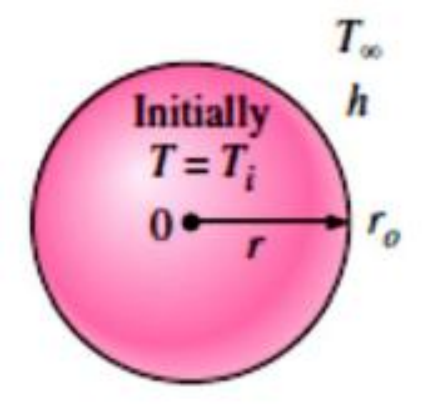
Use the first chart

$$\frac{\alpha\, t}{r_0^2} \cong 0.5$$

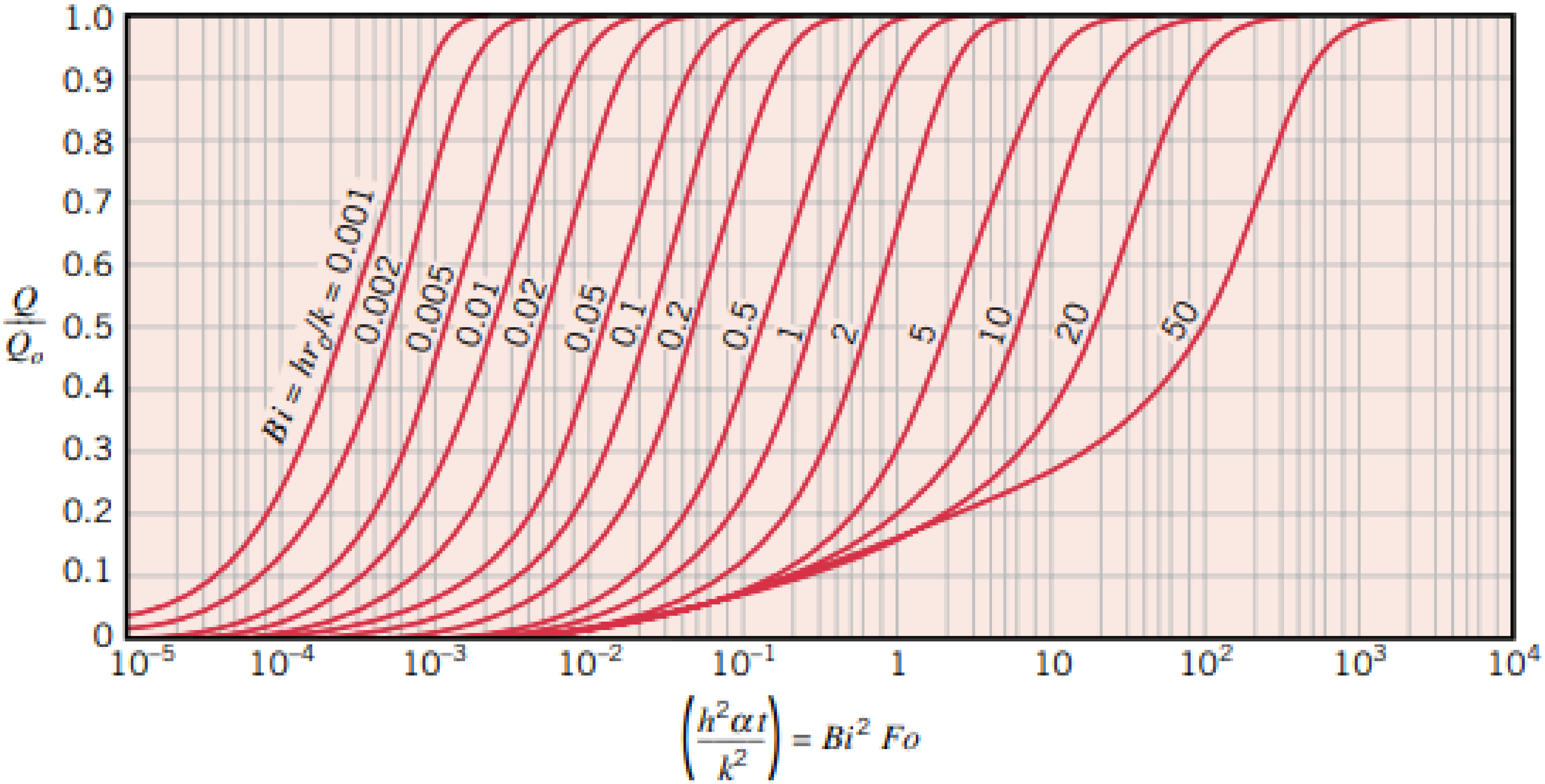
$$t = (0.5)\,\frac{r_0^2}{\alpha} = (0.5)\,\frac{0.01}{1.4\;10^{-5}} = 357.14\ \text{s} = 5.95\ \text{min}$$



The temperature at a position r/r_0 is found using the second chart given here.



Graphical results for the energy transferred from a sphere over the time interval t are presented in third chart given below.



One-term approximation for Heisler charts

It is difficult to read the upper left hand corner of Heissler charts. Approximate solutions can be obtained with one-term approximations of the exact solutions within 2 % accuracy. Define $\lambda_1 \tan(\lambda_1) = Bi$, then:

$$\theta(x,t)_{slab} = \left(\frac{T(x,t) - T_f}{T_i - T_f}\right)_{slab} = C_1 \exp(-\lambda_1^2 Fo) \cos\left(\frac{\lambda_1 x}{L}\right) \quad , \quad Fo > 0.2$$

$$\theta(r,t)_{inf\ cyl} = \left(\frac{T(r,t) - T_f}{T_i - T_f}\right)_{inf\ cyl} = C_1 \exp(-\lambda_1^2 Fo) J_0\left(\frac{\lambda_1 r}{r_0}\right) \quad , \quad Fo > 0.2$$

$$\theta(r,t)_{sphere} = \left(\frac{T(r,t) - T_f}{T_i - T_f}\right)_{sphere} = C_1 \exp(-\lambda_1^2 Fo) \frac{\sin\left(\frac{\lambda_1 r}{r_0}\right)}{\frac{\lambda_1 r}{r_0}} \quad , \quad Fo > 0.2$$

Total heat transfer can be obtained by integration over volume:

$$Q_{max} = m\, c_p\, (T_f - T_i)$$

$$\left(\frac{Q}{Q_{max}}\right)_{slab} = 1 - \theta_{0,slab} \frac{\sin(\lambda_1)}{\lambda_1}$$

See Tables 11-2 and 11-3

$$\left(\frac{Q}{Q_{max}}\right)_{inf\ cyl} = 1 - 2\, \theta_{0,inf\ cyl} \frac{J_1(\lambda_1)}{\lambda_1}$$

$$\left(\frac{Q}{Q_{max}}\right)_{sphere} = 1 - 3\, \theta_{0,sphere} \frac{\sin(\lambda_1) - \lambda_1 \cos(\lambda_1)}{\lambda_1^3}$$

TABLE 11-2 Constants used in the one-term approximation for one-dimensional transient conduction

	Infinite plane wall with thickness 2L ($L_{char} = L$)		Infinite cylinder ($L_{char} = r_0$)		Sphere ($L_{char} = r_0$)	
$Bi = hL_{char}/k$	λ_1 (rad)	C_1	λ_1 (rad)	C_1	λ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1732	1.0049	0.2439	1.0075	0.2989	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2217	1.0082	0.3142	1.0124	0.3852	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3708	1.0173	0.4550	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0160	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6608	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8448	1.0737
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0185	1.1346	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1725	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1795	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7201
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8674
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8921
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5707	1.2733	2.4050	1.6018	3.1415	2.0000

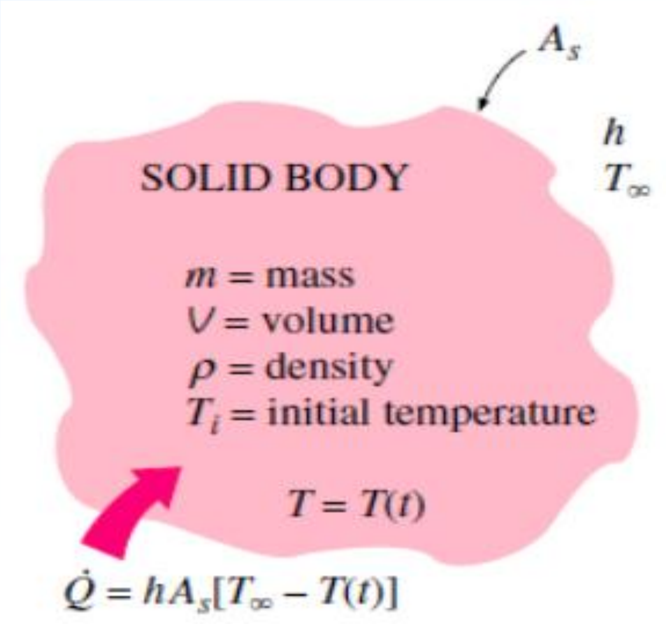
TABLE 11-3 Bessel functions of the first kind

Z	$J_0(Z)$	$J_1(Z)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.5	−0.0484	0.4971
2.6	−0.0968	0.4708
2.7	−0.1424	0.4416
2.8	−0.1850	0.4097
2.9	−0.2243	0.3754
3.0	−0.2601	0.3391

5.2 Lumped-capacity Systems

In transient (time dependent) heat conduction, temperature varies with location and time. Sometimes, the variation with location can be ignored, and the entire solid is assumed to be at a uniform temperature at any given time.

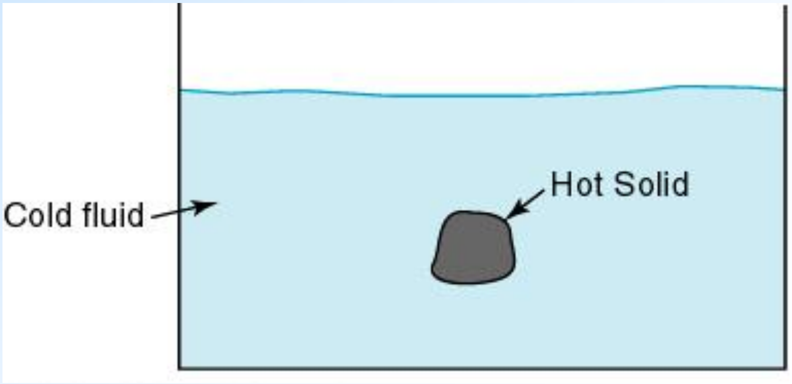
The lumped-capacity system analysis is valid when Biot number is less than 0.1.



$$Bi = \frac{L_c/kA_s}{1/hA_s} = \frac{R_{cond}}{R_{conv}} = \frac{h\,L_c}{k} < 0.1$$

$$L_c = \frac{Volume, V}{Heat\ transfer\ surface\ area, A_s}$$

5.2.1 One Lumped-capacity System with Convection Boundaries



First Law: $\frac{dU}{dt} = \dot{Q}$

$$m\, c_p \frac{dT(t)}{dt} = -\, h\, A\, \left(T(t) - T_f\right)$$

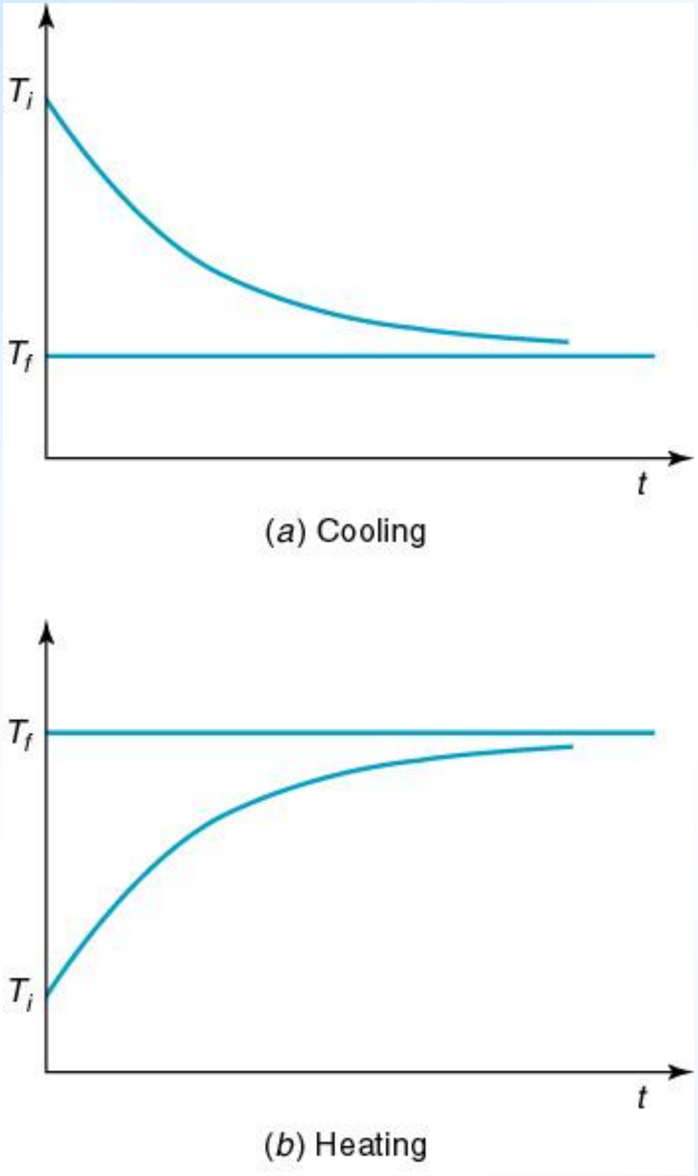
$$\int_{T_i}^{T} \frac{dT}{T - T_f} = -\frac{h\, A}{M\, c_p} \int_0^t dt$$

$$\frac{T(t) - T_f}{T_i - T_f} = exp\left(-\frac{h\, A}{M\, c_p}\, t\right)$$

Substitute $m = \rho\, V$ and $L_c = V/A$

$$\frac{T(t) - T_f}{T_i - T_f} = exp\left(-\frac{h}{\rho\, c_p\, L_c}\, t\right)$$

τ = Time Constant



Change of temperature of a lumped system with time

$$\textit{Time Constant} = \tau = \frac{\rho\, c_p\, L_c}{h}$$

Note that, after about 5 time constants, the temperature of the lumped system reaches almost the steady-state value

Governing equation: $$\frac{d\,T(t)}{d\,t} + \frac{A_s\,h}{\rho\,c_p V}\left(T(t) - T_\infty\right) = 0$$, IC: for t = 0 $T(0) = T_i$

Define: $\theta(t) = \left(T(t) - T_\infty\right)$ and $$\tau = \frac{h\,A_s}{\rho\,c_p V} = \frac{h\,A_s}{m\,c_p} = \frac{h\,\alpha}{k\,L_c}$$ time constant

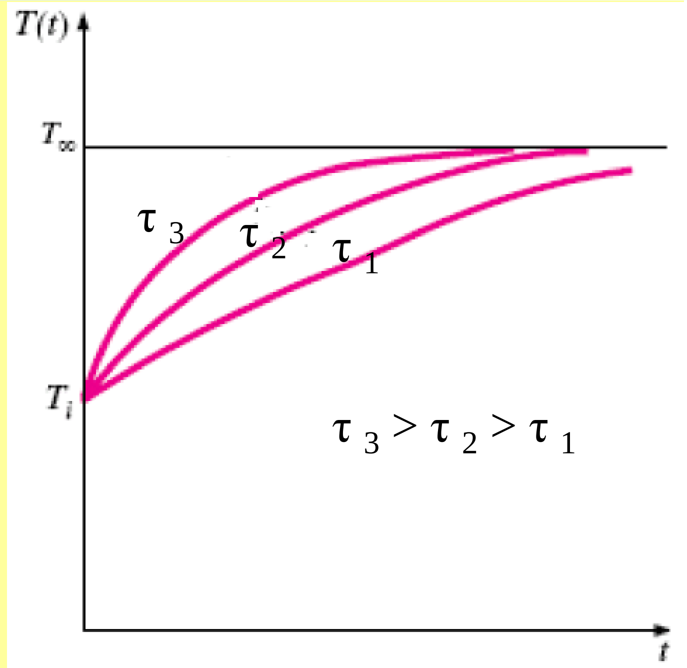
$$\frac{d\,\theta(t)}{d\,t} + \tau\,\theta(t) = 0$$, IC: for t = 0 $\theta(0) = \theta_i = T_i - T_\infty$

Solution: $$\frac{\theta(t)}{\theta_i} = \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\frac{h\,A_s}{\rho\,V\,c_p}t} = e^{-\frac{h\,\alpha}{k\,L_s}t} = e^{-\tau\,t} = e^{-Bi\,Fo}$$

$$Bi = \frac{h\,L_c}{k} = \text{Biot Number}$$ Ratio of resistances, $\frac{R_{cond}}{R_{conv}}$

$$Fo = \frac{\alpha\,t}{L_s^2} = \text{Fourier Number}$$ Ratio of heat conducted to heat stored

$$\frac{\theta(t)}{\theta_i} = \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\frac{h A_s}{\rho V c_p} t} = e^{-\frac{h \alpha}{k L_s} t} = e^{-\tau t} = e^{-Bi\,Fo}$$



- This equation enables us to determine the temperature *T(t) of a* body at time *t, or alternatively, the time t required for the* temperature to reach a specified value, *T(t).*
- The temperature of a body approaches the ambient temperature, T_f or T_∞, exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on.
- A large value of τ *indicates* that the body approaches the ambient temperature in a short time.

The *rate of convection heat transfer between the body* and the ambient can be determined from Newton's law of cooling

$$\dot{Q}(t) = h\,A_s\,\left[T_\infty - T(t)\right] = h\,A_s\,\left(T_\infty - T_i\right)\,e^{-Bi\,Fo} \quad \text{in Watts}$$

The *total heat transfer between the body and the* ambient over the time interval, *0 to t, is simply the* change in the energy content of the body:

$$Q = A_s\,h\int_0^t \left(T(t) - T_\infty\right)dt = m\,c_p\left(T_\infty - T_i\right)\left[1 - e^{-Bi\,Fo}\right] \quad \text{in Joules}$$

The *maximum heat transfer between the body and its surroundings* (when the body temperature reaches T_∞):

$$Q_{max} = m\,c_p\,\left(T_\infty - T_i\right) \quad \text{in Joules}$$

Example 4

A large, pure Aluminum plate, thickness L = 1 inch (2.5 cm), at a uniform temperature T_i = 93.3 °C is suddenly immersed in a well-stirred fluid at temperature T_∞ = 4.4 °C. The convective heat transfer coefficient is h = 96.5 $W/m^2.K$.

Determine the time required for the center of the plate to reach T_0 = 26.7 °C.

k = 20.6 Wm.K, ρ = 2735 kg/m^3, c_p = 837.3 J/kg.K

Check the Biot number first, and use lumped-capacity system analysis if appropriate, i.e., if Bi < 0.1

$$Bi = \frac{h\,L_c}{k} = \frac{h}{k}\,\frac{\text{Volume}}{\text{HT Area}} \cong \frac{h}{k}\,\frac{A\,L}{2\,A} = \frac{96.5}{207.6}\,\frac{0.0254}{2} = 5.9\;10^{-3} < 0.1$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\frac{h A_s}{\rho V c_p} t} = e^{-\frac{h \alpha}{k L_s} t} = e^{-\tau t} = e^{-Bi\,Fo}$$

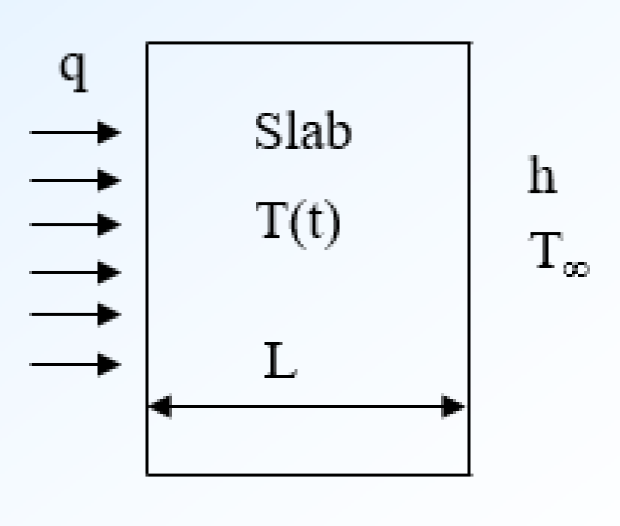
$$\tau = \frac{h A_s}{\rho V c_p} = \frac{h}{\rho c_p L_c} = \frac{96.5}{(2735)\,(837.3)\,\left(0.0254/2\right)} = 3.32\ 10^{-3}$$

Time constant

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \frac{26.7 - 4.4}{93.3 - 4.4} = e^{-(3.32\ 10^{-3})\,t} = 0.25$$

$$t = \frac{\ln\left(\frac{1}{0.25}\right)}{3.32\ 10^{-3}} = 417.8\ \text{s} = 6.96\ \text{min}$$

5.2.2 One Lumped-capacity System with Convection and Prescribed Heat Flux Boundaries



Assumptions:

- Surface areas on both sides of the slab are equal
- Bi < 0.1
- Constant properties

First Law: $$A\,\dot{q} + A\,h\left(T_\infty - T(t)\right) = \rho\, c_p\, A\, L\,\frac{dT(t)}{dt} \quad , \quad t > 0$$

Initial Condition: $T(0) = T_i$

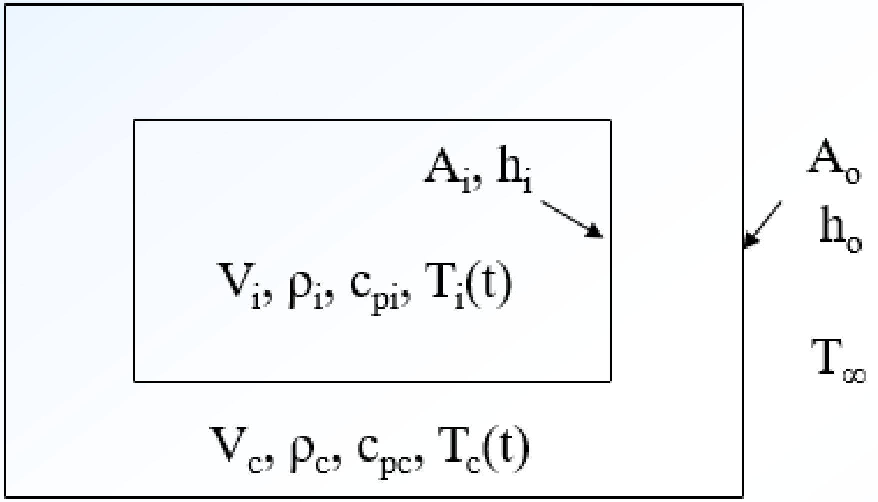
Define:

$$\theta(t) = T(t) - T_\infty \qquad \theta_i = T_i - T_\infty$$

$$\tau = \frac{h}{\rho\, c_p\, L} \qquad \phi = \frac{\dot{q}}{\rho\, c_p\, L}$$

$$\theta(t) = \theta_i\, e^{-\tau t} + (1 - e^{-\tau t})\, \frac{\phi}{\tau}$$

5.2.2 Two Lumped-capacity System with Convection Boundaries



Container, c

Fluid inside, i

Fluid outside, o

Temperatures, T_i and T_c, are functions of time, only

Fluid inside is well stirred. Fluid outside has a constant temperature, T_∞.

Heat transfer areas are A_i and A_o.

Convective heat transfer coefficients are h_i and h_o.

Find the temperature profiles, $T_i(t)$ and $T_c(t)$.

Energy balance equations:

Inner fluid: $$A_i\ h_i\ \left[T_c(t) - T_i(t)\right] = \rho_i\ c_{p\,i}\ V_i\ \frac{dT_i(t)}{dt}$$

For the container: $$A_i\ h_i\ \left[T_i(t) - T_c(t)\right] + A_o\ h_o\ \left[T_\infty - T_c(t)\right] = \rho_c\ c_{p\,c}\ V_c\ \frac{dT_c(t)}{dt}$$

Initial conditions: $T_i(t) = T_c(t) = T_i \quad$ for $t = 0$

Define: $\theta_i(t) = T_i(t) - T_\infty$ $\theta_c(t) = T_c(t) - T_\infty$ $\theta_i = T_o - T_\infty$

$$m_1 = \frac{A_i\ h_i}{\rho_i\ V_i\ c_{p\,i}} \qquad m_2 = \frac{A_i\ h_i}{\rho_c\ V_c\ c_{p\,c}} \qquad m_3 = \frac{A_o\ h_o}{\rho_c\ V_c\ c_{p\,c}}$$

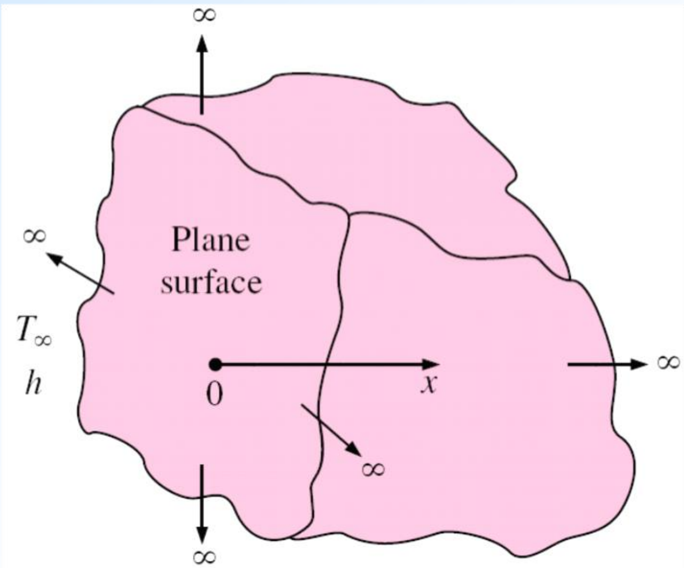
Differential equations become:

$$\frac{d\theta_i(t)}{dt} + m_1\left[\theta_i(t) - \theta_c(t)\right] = 0 \qquad \text{for } t > 0$$

$$\frac{d\theta_c(t)}{dt} + m_2\left[\theta_c(t) - \theta_i(t)\right] + m_3\ \theta_c(t) = 0 \qquad \text{for } t > 0$$

The lengthy solution will not be given here.

5.3 Heat Conduction in Semi-infinite solids



- A semi-infinite solid is an idealized body that has a *single plane surface* and extends to infinity in all directions.

- Assumptions:
  - – constant thermo-physical properties
  - – no internal heat generation
  - – uniform thermal conditions on its exposed surface
  - – initially a uniform temperature T_i *throughout.*

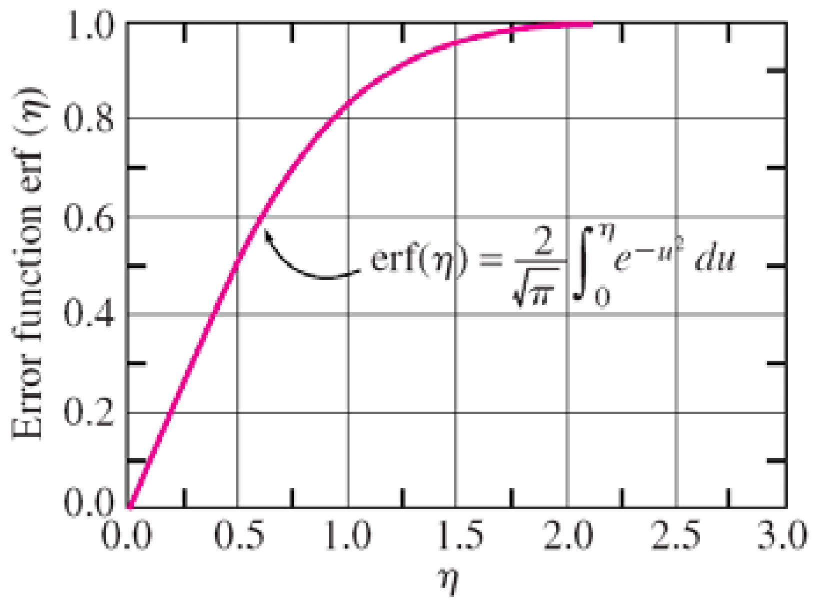
- In this case, heat transfer occurs only in the direction normal to the surface (the *x direction)* ⇨ one-dimensional problem.

Differential Equation: $$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Boundary Conditions: $T(0,t) = T_s$

$T(x \rightarrow \infty,t) = T_i$

Initial Condition: $T(x,0) = T_i$



Solution: $$\frac{(T(t) - T_s)}{(T_i - T_s)} = \frac{2}{\sqrt{\pi}}\int e^{-u^2} du = \mathrm{erf}(\eta) = 1 - \mathrm{erfc}(1 - \eta)$$

where $$\eta = \frac{x}{2\sqrt{\alpha t}}$$ Similarity variable

See the related equations and graphs in the text book for other boundary conditions:

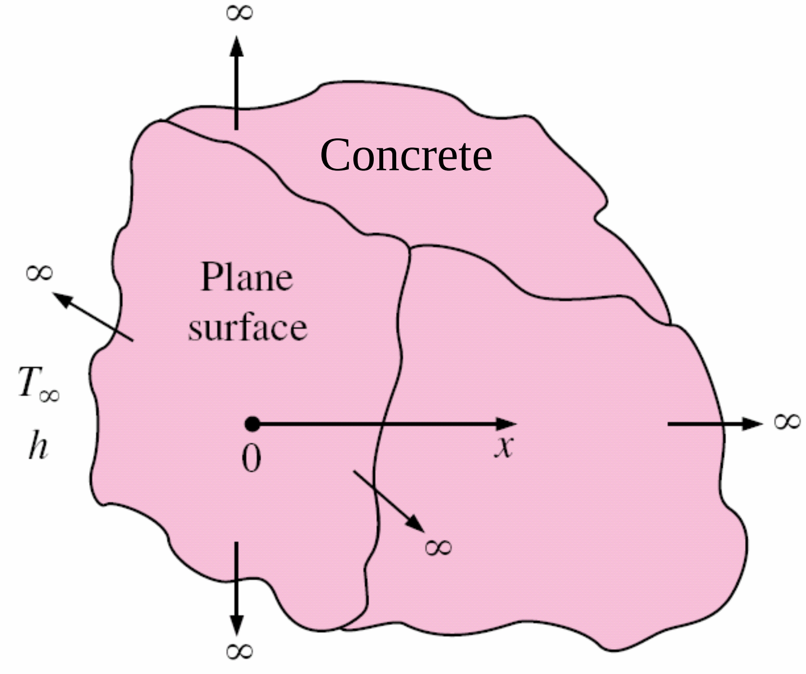
- Constant surface temperature
- Constant surface heat flux
- Convection at the surface

For convection at the surface:

$$\frac{(T(t) - T_s)}{(T_i - T_s)} = 1 - \mathrm{erf}(\eta) - \left[\exp\left(\frac{h\,x}{k} + \frac{h^2 \alpha\, t}{k^2}\right)\right]\left[1 - \mathrm{erf}\left(\eta + \frac{h\sqrt{\alpha\, t}}{k}\right)\right]$$

$$\eta = \frac{x}{2\sqrt{\alpha\, t}}$$

Example 5: Concrete semi-infinite body



Given data:

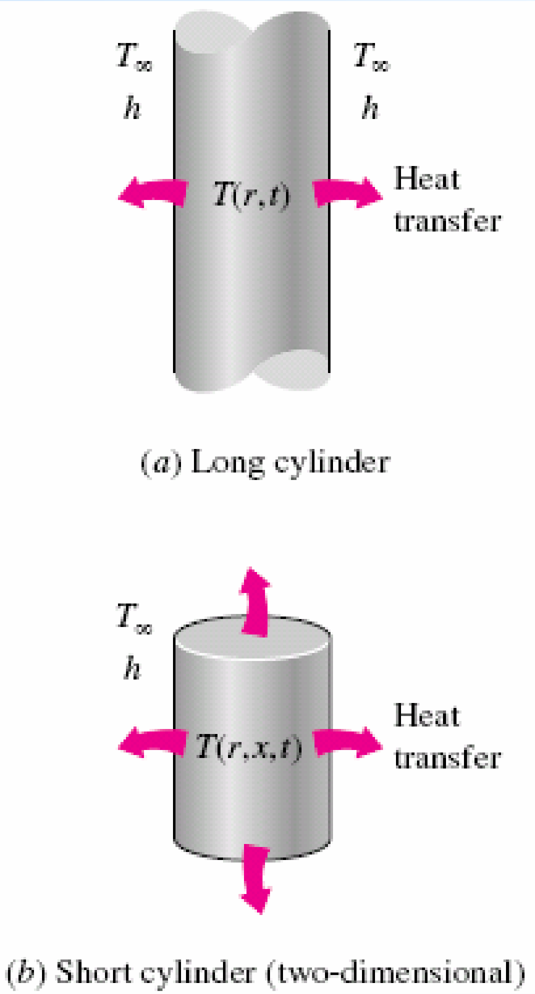
T_i = 54 °C   h = 2.6 W/m^2.K

T_∞ = 10 °C   x = 7 cm

t = 30 minutes

Question: T(x) = ?

5.4 Multi-dimensional Systems – Product Solutions



(*a*) Long cylinder

(*b*) Short cylinder (two-dimensional)

Using a superposition approach called the **product solution, the *one-dimensional heat*** conduction solutions can also be used to construct solutions for some two-dimensional (and even three-dimensional) transient heat conduction problems.

Provided that *all surfaces of the solid are* subjected to convection to the *same fluid* temperature, the *same heat transfer coefficient, h, and the body has no internal heat generation.*

The solution can be generalized as follows:

The solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.

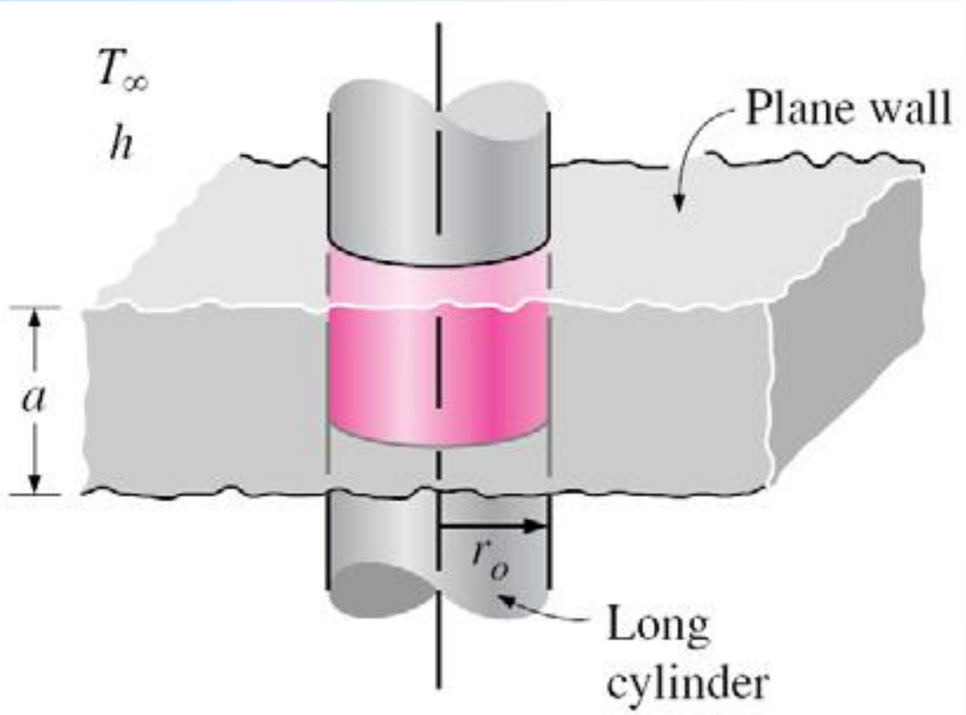
For convenience, the one-dimensional solutions are denoted by

Plane Wall (slab) $$\left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{plane\ wall}$$

Infinite Cylinder $$\left.\frac{\left(T(t,r) - T_\infty\right)}{(T_i - T_\infty)}\right|_{\text{inf}.\ cyl.}$$

Semi-infinite body $$\left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{semi-\text{inf}.\ \text{body}}$$

Example 6: Short Cylinder

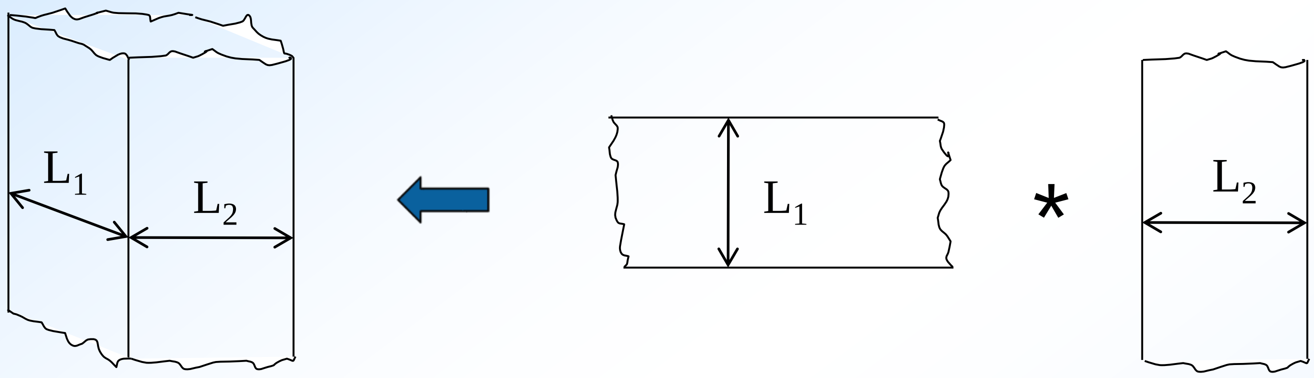


- Height *a and radius* r_o.
- Initially uniform temperature T_i.
- No heat generation
- At time $t = 0$:
  - – convection T_∞
  - – heat transfer coefficient , h

Product solution:

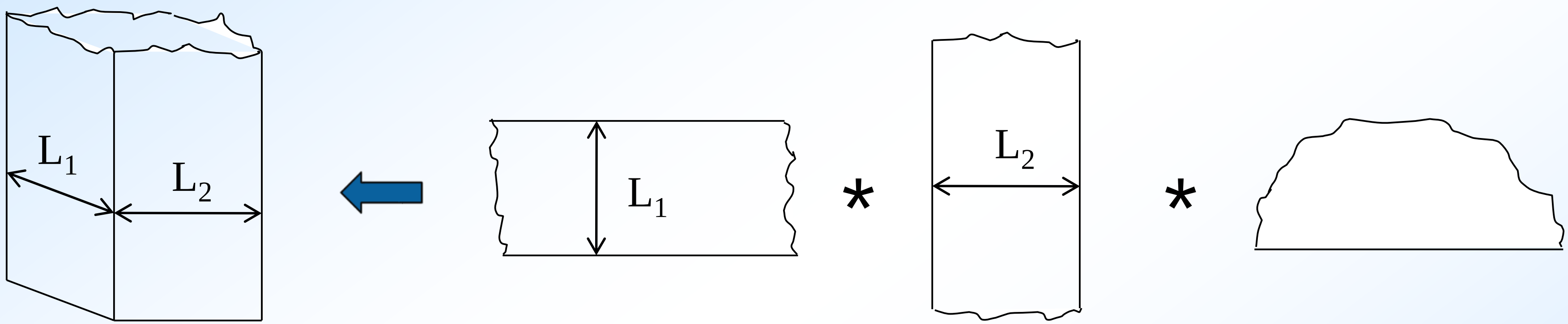
$$\left.\frac{\left(T(t,r,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{short\ cyl.} = \left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{plane\ wall} * \left.\frac{\left(T(t,r) - T_\infty\right)}{(T_i - T_\infty)}\right|_{\mathrm{inf.}\ cyl.}$$

Infinite rectangular bar:



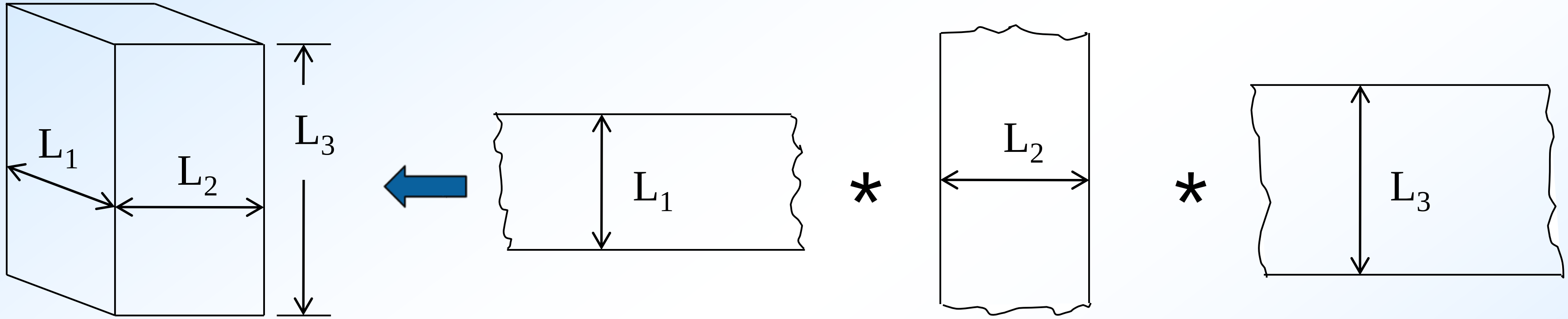
$$\left.\frac{\left(T(t,x) - T_\infty\right)}{\left(T_i - T_\infty\right)}\right|_{inf.\ bar} = \left.\frac{\left(T(t,x) - T_\infty\right)}{\left(T_i - T_\infty\right)}\right|_{plate,\ L_1} * \left.\frac{\left(T(t,x) - T_\infty\right)}{\left(T_i - T_\infty\right)}\right|_{plate,\ L_2}$$

Semi-infinite rectangular bar:



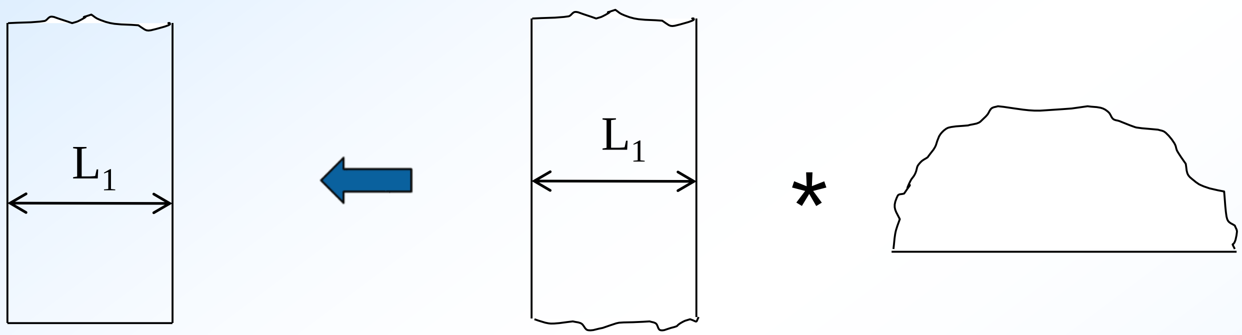
$$\left.\frac{\left(T(t,x) - T_{\infty}\right)}{\left(T_i - T_{\infty}\right)}\right|_{semi-\inf.\,bar} = \left.\frac{\left(T(t,x) - T_{\infty}\right)}{\left(T_i - T_{\infty}\right)}\right|_{plate,\,L_1} * \left.\frac{\left(T(t,x) - T_{\infty}\right)}{\left(T_i - T_{\infty}\right)}\right|_{plate,\,L_2} * \left.\frac{\left(T(t,x) - T_{\infty}\right)}{\left(T_i - T_{\infty}\right)}\right|_{semi-\inf.\,body}$$

Rectangular parallelepiped:



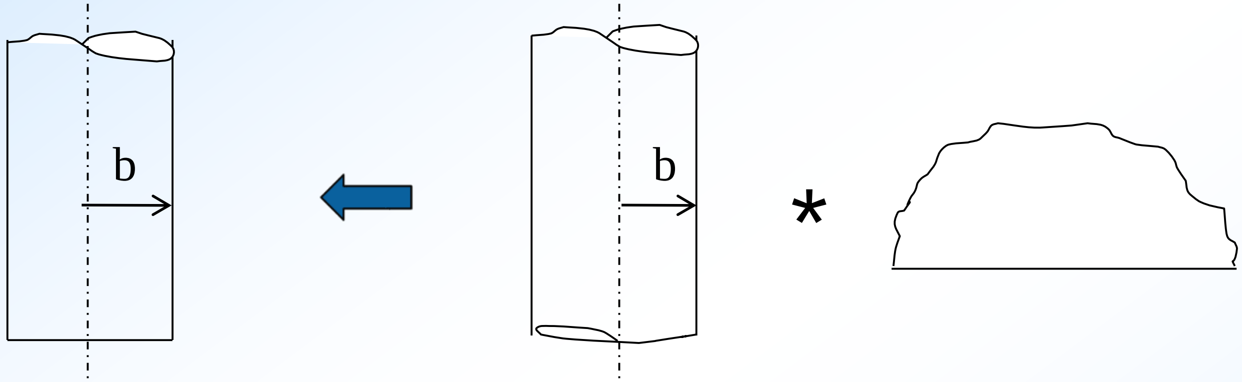
$$\left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{rec.\,pp} = \left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{plate,\,L_1} * \left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{plate,\,L_2} * \left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{plate,\,L_3}$$

Semi-infinite slab:



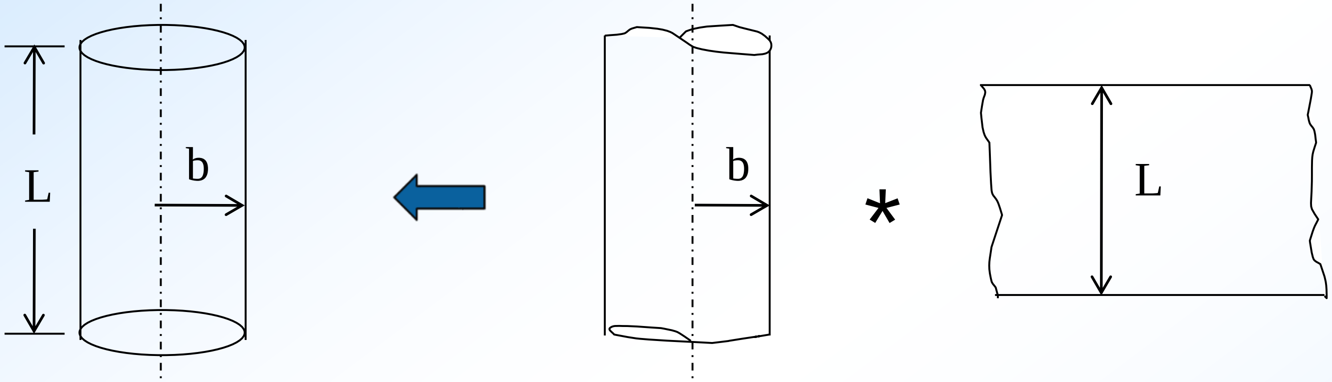
$$\left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{semi-\inf.\ slab} = \left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{plate,\ L_1} * \left.\frac{\left(T(t,x) - T_\infty\right)}{(T_i - T_\infty)}\right|_{semi-\inf.\ body}$$

Semi-infinite cylinder:



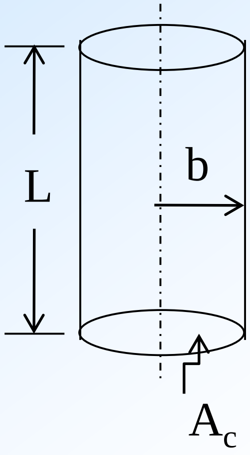
$$\left.\frac{\left(T(t,r) - T_{\infty}\right)}{(T_i - T_{\infty})}\right|_{semi-\inf.\,cyl.} = \left.\frac{\left(T(t,r) - T_{\infty}\right)}{(T_i - T_{\infty})}\right|_{\inf.\,cyl.} * \left.\frac{\left(T(t,x) - T_{\infty}\right)}{(T_i - T_{\infty})}\right|_{semi-\inf.\,body}$$

Finite (short) cylinder:



$$\left.\frac{\left(T(t,r) - T_{\infty}\right)}{(T_i - T_{\infty})}\right|_{finite\ cyl.} = \left.\frac{\left(T(t,r) - T_{\infty}\right)}{(T_i - T_{\infty})}\right|_{\inf.\ cyl.} * \left.\frac{\left(T(t,x) - T_{\infty}\right)}{(T_i - T_{\infty})}\right|_{plate,\ L}$$

Example 7



Hot metals are quenched in cold fluids to change the material properties. Consider a long 7.5 diameter cylinder of 316 stainless steel that is taken out of the furnace at 500 C and plunged into a cold bath at 25 C. The convective heat transfer coefficient is 1000 W/m^2.K

(a) Determine the centerline temperature of the cylinder 90 s after it is quenched

(b) Determine the surface temperature of the cylinder 5 min after it is quenched

(c) Determine the time required for the centerline temperature to reach 50 C.