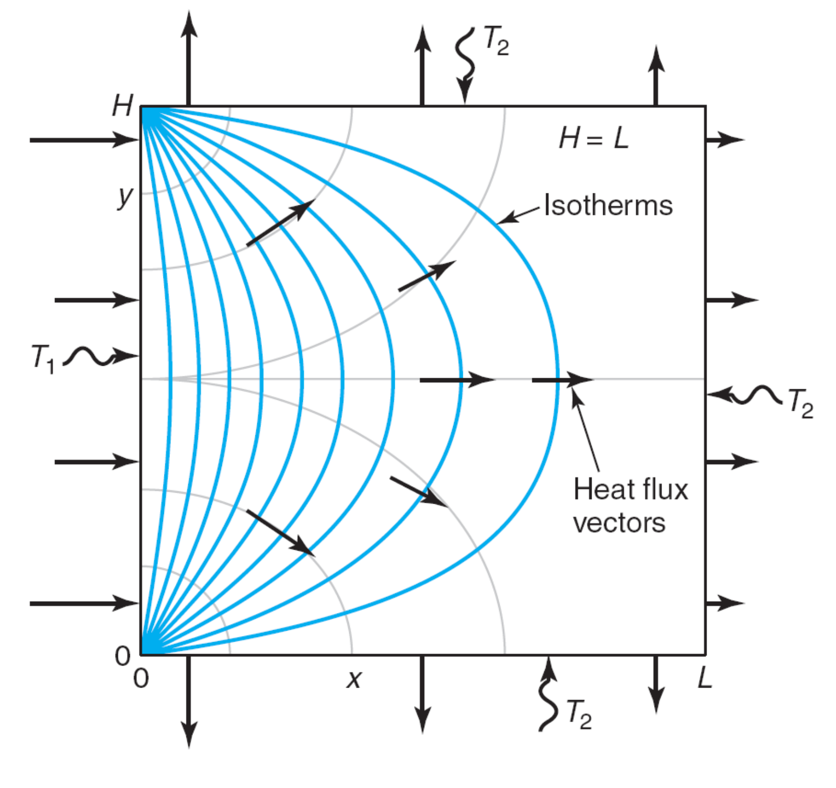
4. Steady, Two-dimensional Heat Conduction

The differential equation of heat conduction in Cartesian coordinates with constant k and no heat generation:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$



There are similar equations in cylindrical and spherical coordinates (for a cylinder and a sphere)

Note the characteristics:

- Partial derivatives
- Requires four boundary conditions

The four boundary conditions for the given Figure, for instance, are:

$$T(0,y) = T_1 \quad T(L,y) = T_2$$

$$T(x,0) = T_2 \quad T(x,H) = T_2$$

There are three techniques to solve this partial differential equation:

1. Analytical;
2. Graphical; and
3. Numerical.

The analytical (exact) solution using a method called separation of variables is:

$$\frac{T(x,y) - T_2}{T_1 - T_2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{sinh[(2n+1)\,\pi\,(L-x)/H]}{sinh[(2\,n+1)(\pi\,L/H]} \; \frac{sin[(2n+1)\,(\pi\,y/H]}{2\,n+1}$$

This is the temperature profile (distribution), T(x,y), in the solid. To find the heat flow rate, Q_x and Q_y, apply Fourier's law.

The analytical (exact) solution is beyond our scope here.

In practice, the heat flow at the boundaries of the region is of interest:

$$\dot{q}_x = -k\frac{\partial T}{\partial x} \quad \text{and} \quad \dot{q}_y = -k\frac{\partial T}{\partial y}$$

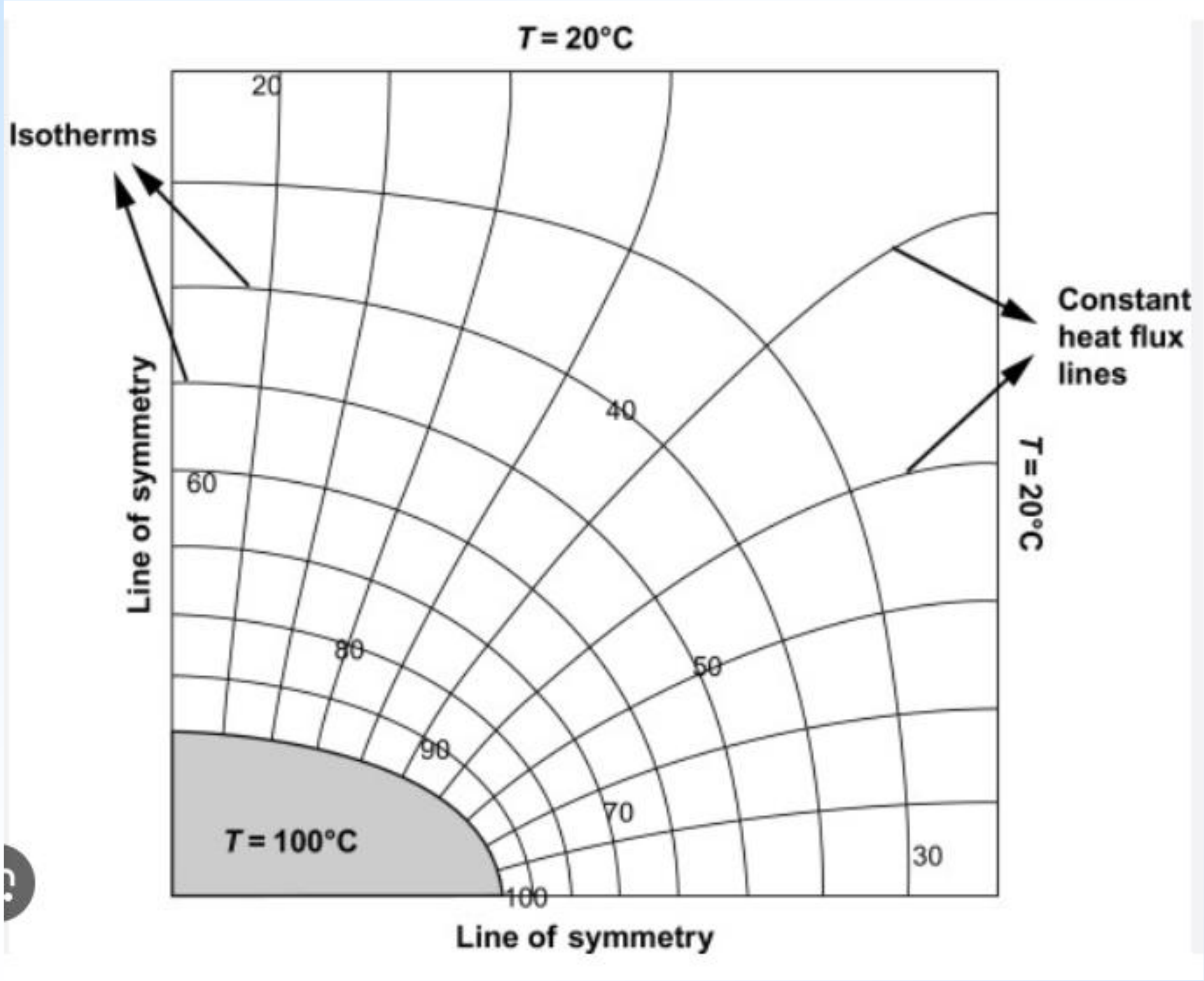
The total heat flux is the vectorial sum of q_x and q_y. The total heat flow vector is perpendicular to constant temperature lines (called isotherms) in the material. Using this, one can find the total heat flow by constructing curvilinear-square plots of isotherms and heat flow lanes. This is the basis of graphical solutions.

Other, more complex geometries, can be handled by numerical solutions, which change the partial differential equation to a set of algebraic equations to be solved with matrix handling routines.

There is a short-cut procedure to find the heat flow rate (not the temperature profile) when thermal resistance concept can be used, i.e., constant k, steady state, no heat generation, finite regions, plus temperatures are specified at the boundaries.

4.1 Graphical Analysis

In the given two dimensional geometry, draw constant-temperature lines (called isotherms) as many as possible (more the better) and heat flow lines (called adiabats) approximately perpendicular to the isotherms and forming approximate squares. An example is shown in the Figure.



In the Figure, 10 °C steps are used for the isotherms.

Adiabats (heat flow lines) are perpendicular to the isotherms and form squares, approximately

Question: Q = ?

$$\dot{Q} = M\,Q_i$$

Number of heat flow lanes

Heat flow rate through each lane

$$\dot{Q}_i = k\,A_i\,\frac{\Delta T_i}{\Delta y_i}$$

$A_i = w\,\Delta x_i$

w is the width of the region

$\Delta x_i \cong \Delta y_i$

$$\dot{Q}_i = k\,w\,\Delta T_i$$

$$\Delta T_i = \frac{\Delta T_{overall}}{N}$$ where N = Number of equal temperature intervals

$$\dot{Q}_i = k\,w\,\Delta T_i = k\,w\,\frac{\Delta T_{overall}}{N} \quad \Rightarrow \quad \frac{\dot{Q}}{w} = \frac{M}{N}\,k\,\Delta T_{overall}$$

Define Shape Factor: $$S = \frac{M}{N} = \frac{\text{Number of heat flow lanes}}{\text{Number of equal temperature intervals}}$$

$$\frac{\dot{Q}}{w} = S\,k\,\Delta T_{overall}$$

4.2 Conduction Shape Factor

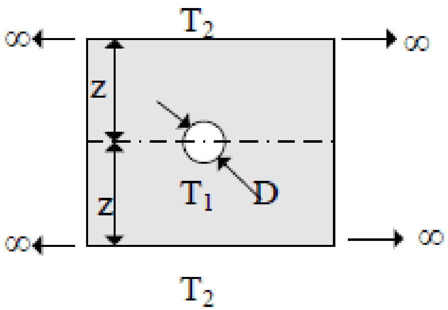
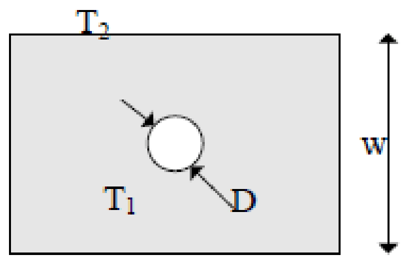
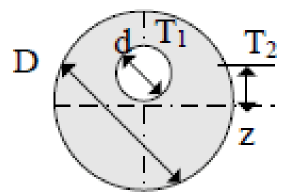
Two-dimensional heat transfer in a medium bounded by two isothermal surfaces at T_1 and T_2 may be represented in terms of a conduction shape factor, *S.*

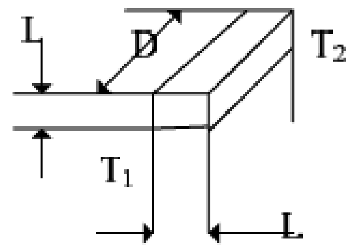
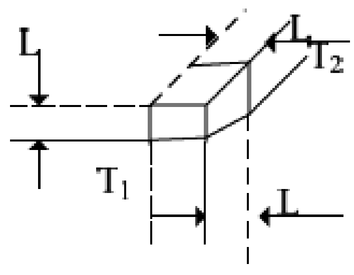
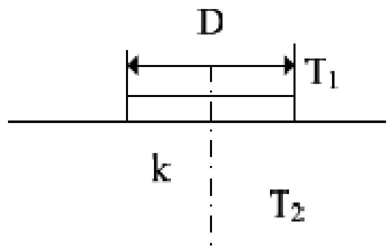
$$\dot{Q} = S\,k\,(T_1 - T_2) \qquad R_{thermal} = \frac{1}{k\,S}$$

Plane wall: $$\dot{Q} = k\,A\,\frac{\Delta T}{\Delta x} \quad \Rightarrow \quad S = \frac{A}{\Delta x} = \frac{A}{L}$$

Hollow cylinder: $$\dot{Q} = \frac{2\,\pi\,L\,k}{\ln(r_2/r_1)}\,\Delta T \quad \Rightarrow \quad S = \frac{2\,\pi\,L}{\ln(r_2/r_1)}$$

Hollow sphere: $$\dot{Q} = \frac{4\,\pi\,r_1\,r_2\,k}{r_2 - r_1}\,\Delta T \quad \Rightarrow \quad S = \frac{4\,\pi\,r_1\,r_2}{r_2 - r_1}$$

Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		z>>D/2 L>>2	$\dfrac{2\pi L}{\ln(8z/\pi D)}$
Circular cylinder of length L centered in a square solid of equal length		W>D L>>w	$\dfrac{2\pi L}{\ln(1.08w/D)}$
Eccentric circular cylinder of length L in a cylinder of equal length		D>d L>>D	$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2+d^2-4z^2}{2Dd}\right)}$

Conduction through the edge of adjoining walls		$D > L/5$	$0.54D$
Conduction through corners of three walls with a temperature difference of ΔT_{1-2} across the walls		$L <<$ length and width of wall	$0.15L$
Disk of diameter D and T1 on a semi finite medium of thermal conductivity k and T_2		None	$2D$

Example 1

A small furnace, 50 x 60 x 70 cm on the inside, is constructed of fireclay brick (k = 1 W/m.K) with a wall thickness of 10 cm. The inner and outer surfaces are maintained at 500 °C and 50 °C, respectively. Calculate the heat loss.

$$\dot{Q}_{total} = \dot{Q}_{corners} + \dot{Q}_{edges} + \dot{Q}_{walls}$$

$$\dot{Q}_{corners} = (8)\,(0.15)\,(0.1)\,(1.04)\,(500 - 50) = 56 \text{ W}$$

$$\dot{Q}_{edges} = (4)\,(1.04)\,(500 - 50)\,(0.54)\,\left[(0.5) + (0.6) + (0.7)\right] = 1820 \text{ W}$$

$$\dot{Q}_{edges} = (2)\,(1.04)\,(500 - 50)\left[\frac{(0.5)\,(0.6) + (0.6)\,(0.7) + (0.5)\,(0.7)}{0.1}\right] = 10015 \text{ W}$$

$$\dot{Q}_{total} = 0.056 + 1.82 + 10 \cong 12 \text{ kW}$$

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$Q_{total} = Q_{corners} + Q_{edges} + Q_{walls}$

Show that $Q_{total} = 0.056 + 1.82 + 10 = 12$ kW