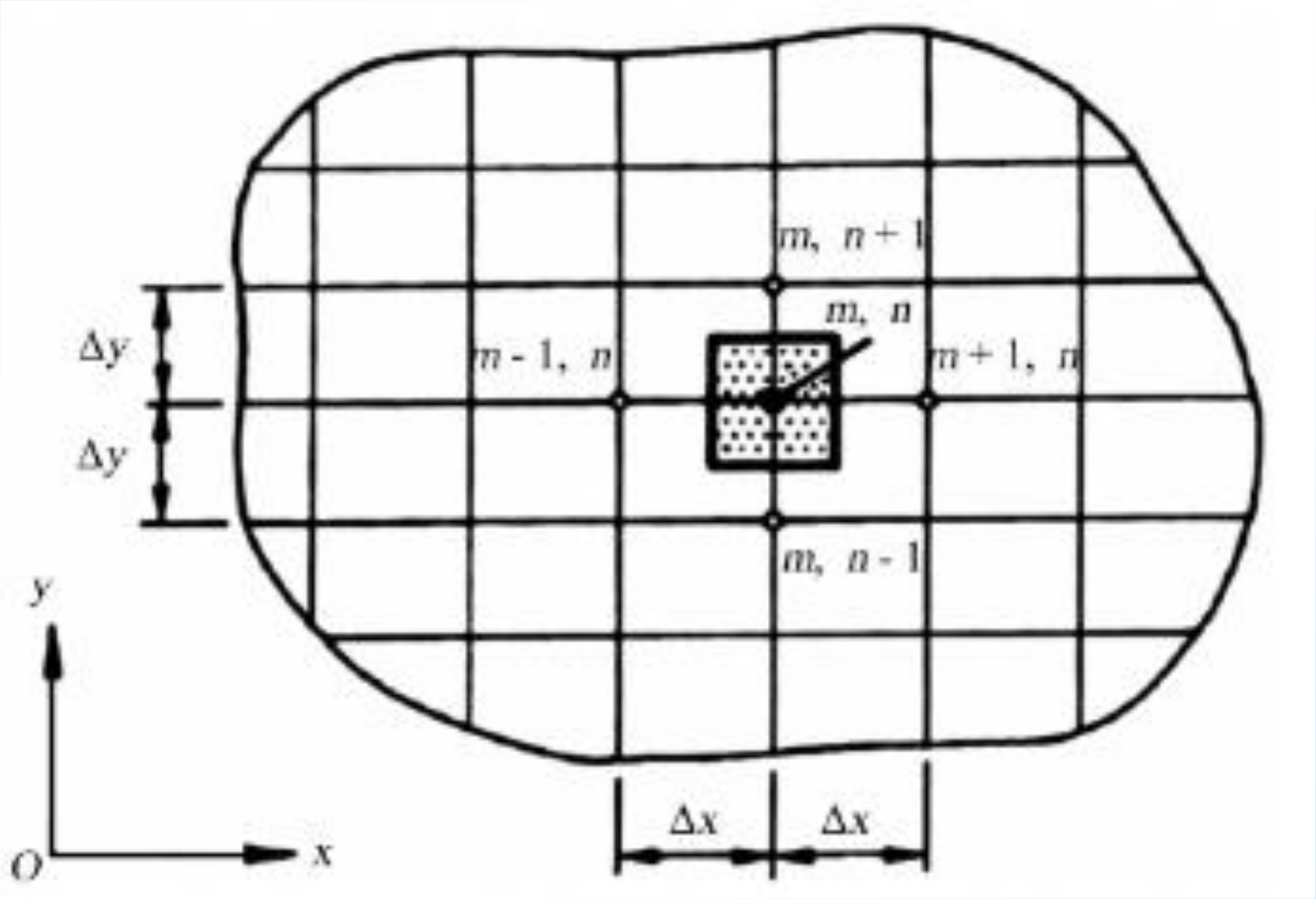
6. Heat Conduction – Numerical Methods of Solution

For solids having complicated boundary conditions and geometries, Numerical solution are extensively used.

The finite-difference method will be discussed here. Another numerical schemes such as finite-element and finite-voluıme methods which offer some advantages over the finite-difference method will not be discussed.

The finite-difference method involves the approximation of partial differential equations with a set of algebraic equations for temperatures at chosen nodal points defined over the region of interest.

6.1 Two-dimensional, Steady State, Constant Properties



$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{\dot{e}_{gen}(x,y)}{k} = 0 \quad \text{in region R}$$

The exact solution requires four boundary conditions. What would you do when the boundaries are irregular as in the figure?

Finite difference approximation: Replace all the derivatives (in the differential equation as well as the boundary conditions) with differences satisfied at the nodal points of the mash. Central differences are preferably used for higher accuracy.

$$\left.\frac{\partial T(x,y)}{\partial x}\right|_{m+\frac{1}{2},n} \cong \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T(x,y)}{\partial x}\right|_{m-\frac{1}{2},n} \cong \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T^2(x,y)}{\partial x^2}\right|_{m,n} \cong \frac{\left.\frac{\partial T(x,y)}{\partial x}\right|_{m+\frac{1}{2},n} - \left.\frac{\partial T(x,y)}{\partial x}\right|_{m-\frac{1}{2},n}}{\Delta x}$$

$$\left.\frac{\partial T^2(x,y)}{\partial x^2}\right|_{m,n} \cong \frac{T_{m-1,n} + T_{m+1,n} - 2\,T_{m,n}}{\Delta x^2}$$

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$$\left.\frac{\partial T^2(x,y)}{\partial y^2}\right|_{m,n} \cong \frac{T_{m,n-1} + T_{m,n+1} - 2\,T_{m,n}}{\Delta y^2}$$

Note the use of central differences. Forward and backward differences may also be used.

Substitute into the differential equation. If $\Delta x = \Delta y = L$

$$\boxed{T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4\,T_{m,n} + \frac{\dot{e}_{m,n}\,L^2}{k} = 0}$$

This is the finite difference representation of the heat conduction equation for the node (m,n), applicable if (m,n) is in the region, not on a boundary.

$e_{m,n}$ is the heat generation per unit volume at (m,n).

The same equation can be derived applying the first law to the shaded area, i.e.:

$$\left\{\begin{matrix}\text{Heat} \\ \text{Generated}\end{matrix}\right\} = \left\{\begin{matrix}\text{Heat} \\ \text{Flowing out}\end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix}\text{Heat} \\ \text{Flowing in}\end{matrix}\right\} + \left\{\begin{matrix}\text{Heat} \\ \text{Generated}\end{matrix}\right\} = 0$$

$$\left\{\begin{matrix}\text{Heat} \\ \text{Flowing in}\end{matrix}\right\} = k\,\Delta y\,w\,\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\,\Delta y\,w\,\frac{T_{m+1,n} - T_{m,n}}{\Delta x} +$$

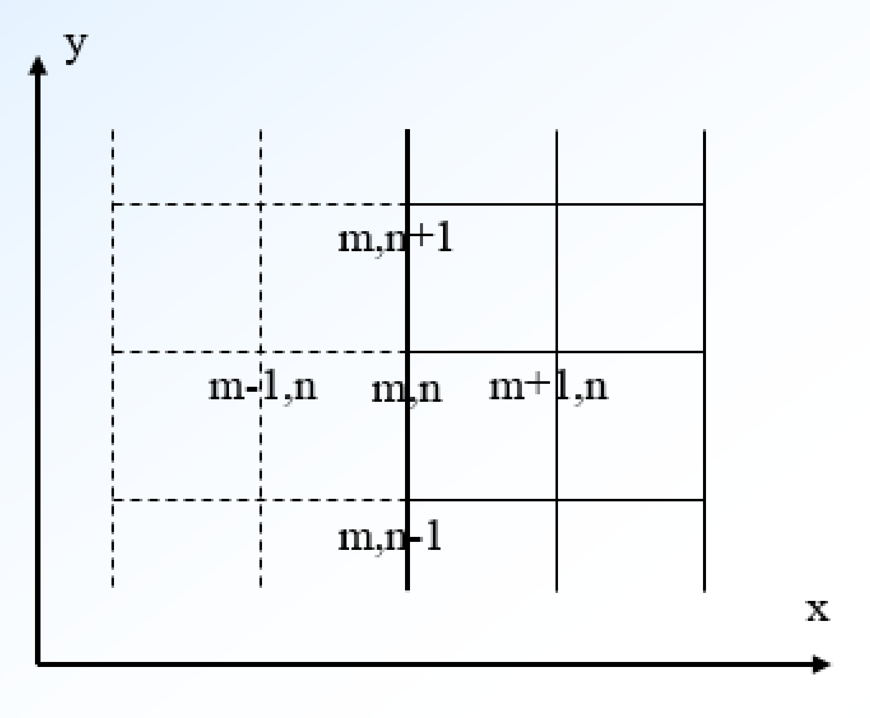
$$k\,\Delta x\,w\,\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k\,\Delta x\,w\,\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\left\{\begin{matrix}\text{Heat} \\ \text{Generated}\end{matrix}\right\} = \dot{e}_{m,n}\,\Delta x\,\Delta y\,w$$

Substitute to find the same equation when $\Delta x = \Delta y$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4\,T_{m,n} + \frac{\dot{e}_{m,n}\,L^2}{k} = 0$$

6.1.1 Node on an adabatic (insulated) boundary



Consider the node (m.n) on such a boundary. The heat flux in the x-direction is zero. i.e., $q_x = 0$. Therefore

$$\frac{\partial T}{\partial x} = 0 \qquad \text{or} \qquad T_{m+1,n} = T_{m-1,n}$$

where the imaginary mode (m-1,n) is the mirror image of (m+1,n)

The differential equation becomes:

$$2\,T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4\,T_{m,n} + \frac{\dot{e}_{m,n}\,L^2}{k} = 0$$

Similarly, if the adiabatic boundary is parallel to the x-axis (no heat flow in the y direction), then

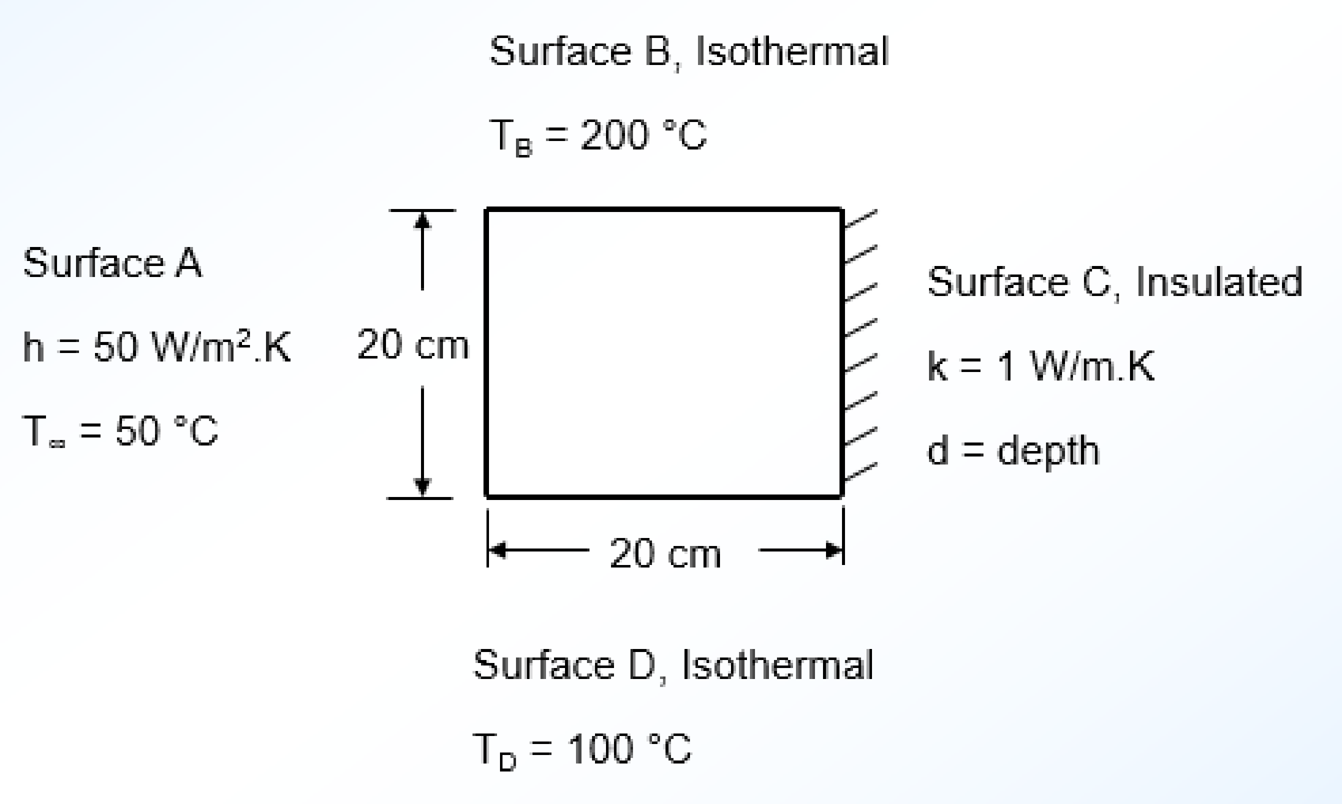
$$T_{m-1,n} + T_{m+1,n} + 2\,T_{m,n+1} - 4\,T_{m,n} + \frac{\dot{e}_{m,n}\,L^2}{k} = 0$$

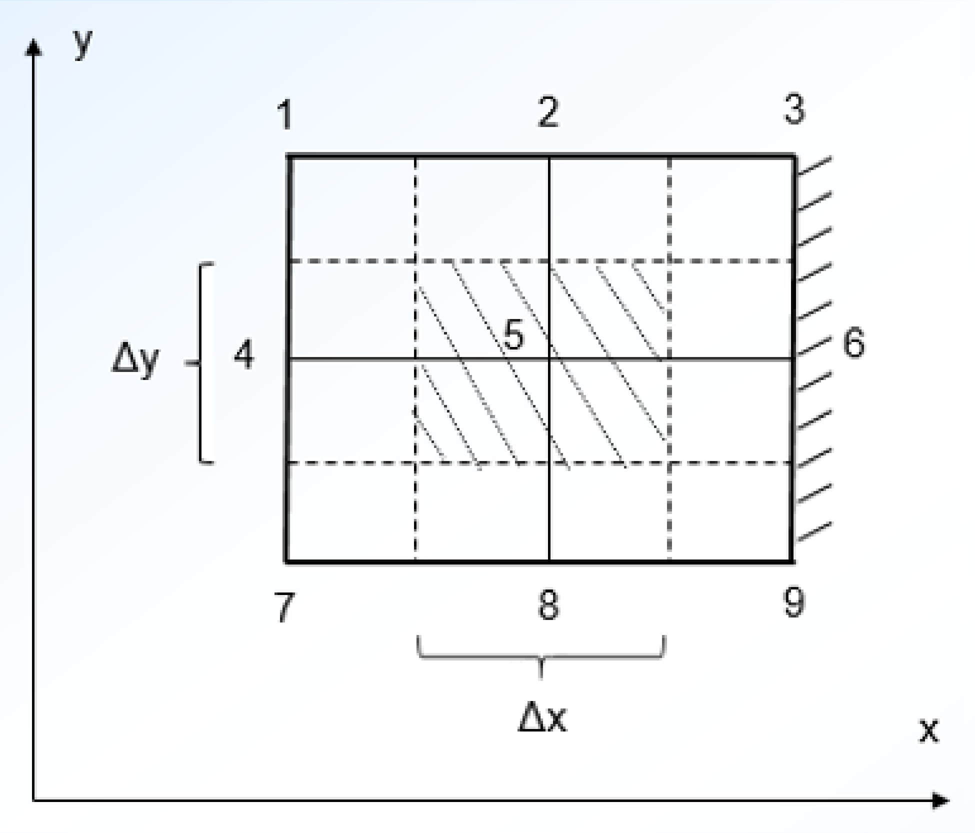
If the node (m,n) is at the intersection of two adiabatic boundaries, then

$$2\,T_{m+1,n} + T_{m,n+1} - 4\,T_{m,n} + \frac{\dot{e}_{m,n}\,L^2}{k} = 0$$

Example 1

Determine the steady-state temperature distribution and heat flow rates from all four surfaces of the two-dimensional solid shown in the Figure.





$\Delta x = \Delta y = L = 10$ cm

$T_1 = T_2 = T_3 = T_B = 200$ °C

$T_7 = T_8 = T_9 = T_D = 100$ °C

$T_4 = T_5 = T_6 = ?$

T_4 is on a convection boundary.

T_5 is an interior node.

T_6 is on an isulated surface.

$$\frac{h\,L}{k} = \frac{(50)\,(0.1)}{1} = 5$$

$$\frac{1}{2}\left(T_1 + T_7\right) + T_5 - \frac{h\,L}{k}\,T_\infty - \left(2 + \frac{h\,L}{k}\right) T_4 = 0$$

$$\boxed{-7\,T_4 + T_5 + 400 = 0}$$

$$T_4 + T_2 + T_6 + T_8 - 4\,T_5 = 0$$

$$\boxed{T_4 - 4\,T_5 + T_6 + 300 = 0}$$

$$\frac{1}{2}\left(T_3 + T_9\right) + T_5 - 2\,T_6 = 0$$

$$\boxed{T_5 - 2\,T_6 + 150 = 0}$$

Solve these three equations with three unknowns simultaneously and find:

$$T_4 = 75.5\ ^\circ C \qquad T_5 = 128.7\ ^\circ C \qquad T_6 = 139.4\ ^\circ C$$

Heat flow rate per unit depth at surface A (into the solid):

$$\frac{Q_A}{d} = h\,\frac{\Delta y}{2}\left(T_\infty - T_1\right) + h\,\Delta y\left(T_\infty - T_4\right) + h\,\Delta y\,\frac{T_\infty - T_7}{2} = -\,627.5\ \text{W/m}$$

Heat flow rate per unit depth at surface B (into the solid):

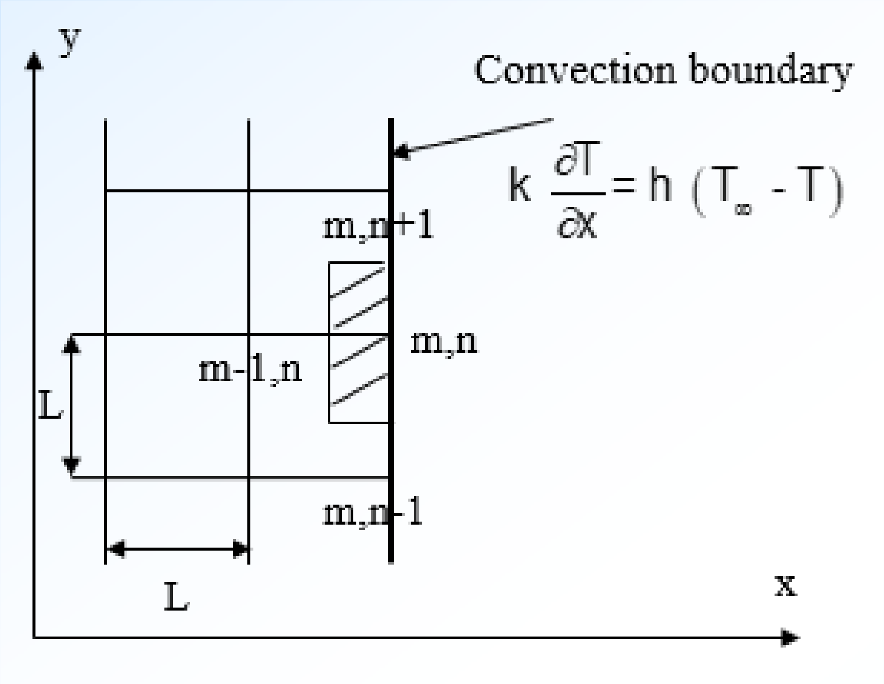
$$\frac{Q_B}{d} = k\,\frac{\Delta x}{2}\,\frac{T_1 - T_4}{\Delta y} + k\,\Delta x\,\frac{T_2 - T_5}{\Delta y} + k\,\Delta x\,\frac{T_3 - T_6}{\Delta y} + h\,\frac{\Delta y}{2}\left(T_1 - T_\infty\right) = 538.8\ \text{W/m}$$

Heat flow rate per unit depth at surface C = 0 (insulated)

Heat flow rate per unit depth at surface D (into the solid):

$$\frac{Q_d}{d} = k\,\frac{\Delta x}{2}\,\frac{T_9 - T_6}{\Delta y} + k\,\Delta x\,\frac{T_8 - T_5}{\Delta y} + k\,\frac{\Delta x}{2}\,\frac{T_7 - T_4}{\Delta y} + h\,\frac{\Delta y}{2}\left(T_7 - T_\infty\right) = 88.8\ \text{W/m}$$

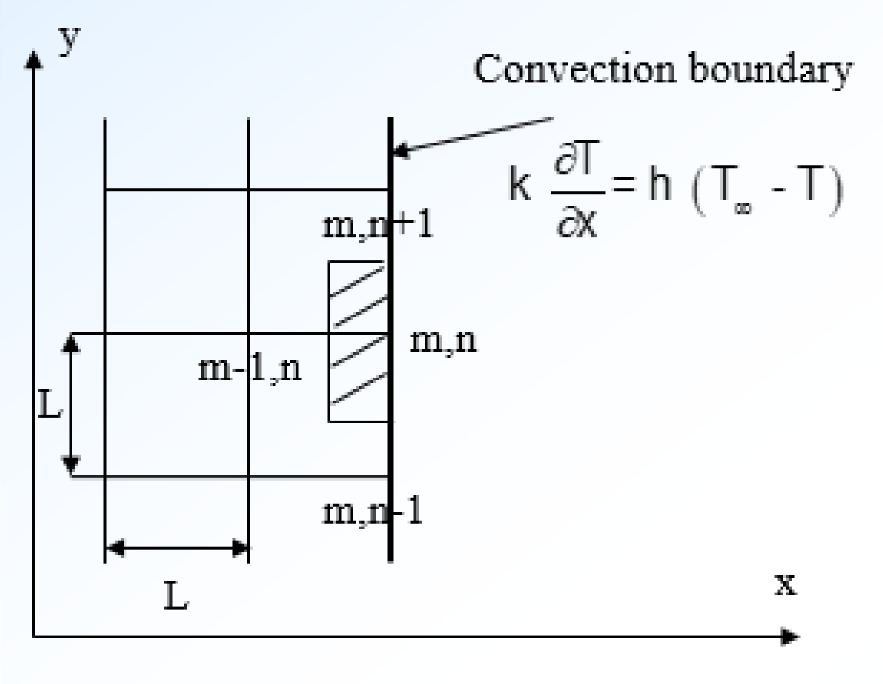
6.1.2 Node on a convection boundary



One can find the difference equation for the node (m,n) by writing an energy balance equation on the shaded volume element (with unit depth).

A set of algebraic equations are written for each node in the region and its boundary and simultaneous solution gives the temperatures at each node.

To calculate the heat flux, the Fourier's Law (or Newton's cooling law) is applied with derivatives replaced with a differences.



Derive this equation:

$$2\,T_{m-1,n} + T_{m,n-1} + T_{m,n+1} - \left(4 + \frac{2\,h\,L}{k}\right) T_{m,n} + \frac{2\,h\,L\,T_\infty}{k} + \frac{\dot{e}_{gen}\,L^2}{k} = 0$$