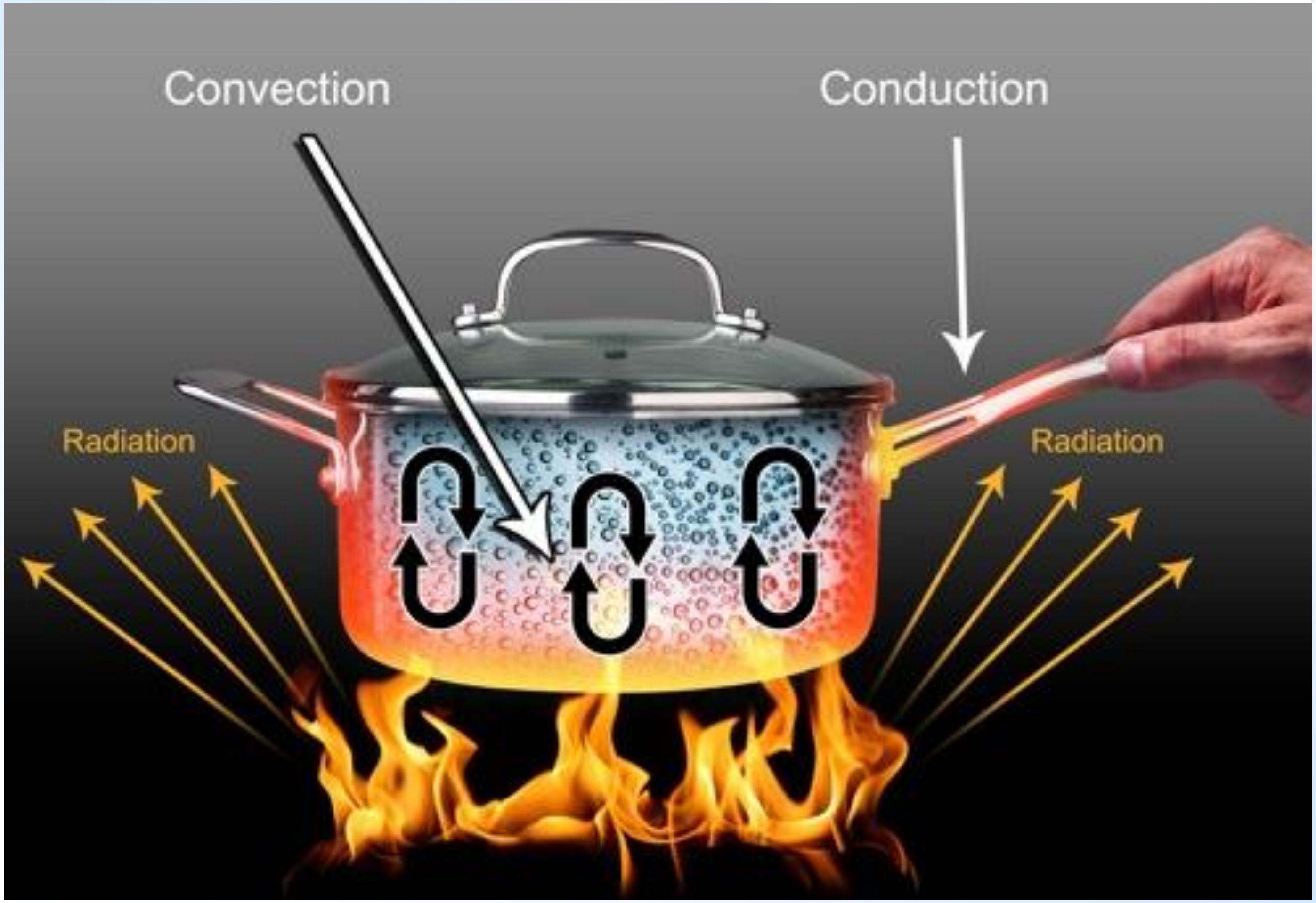
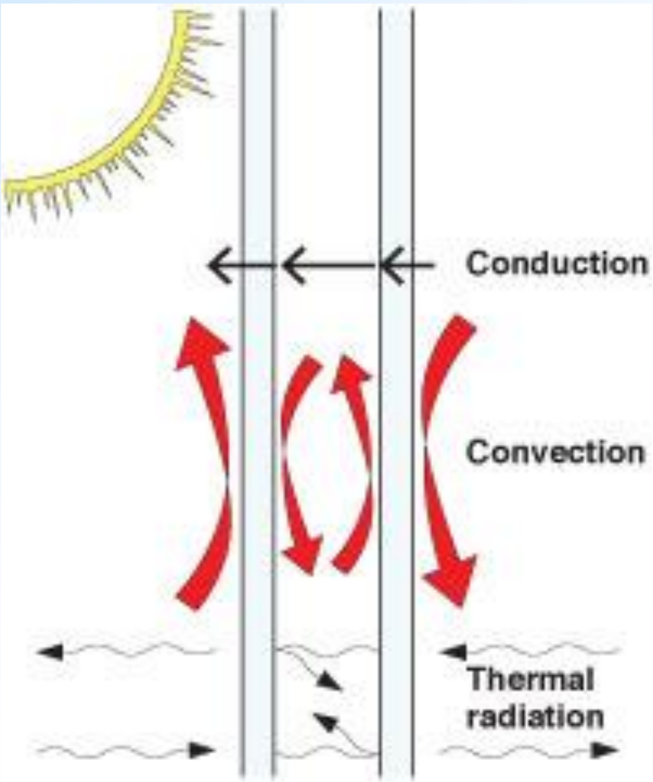
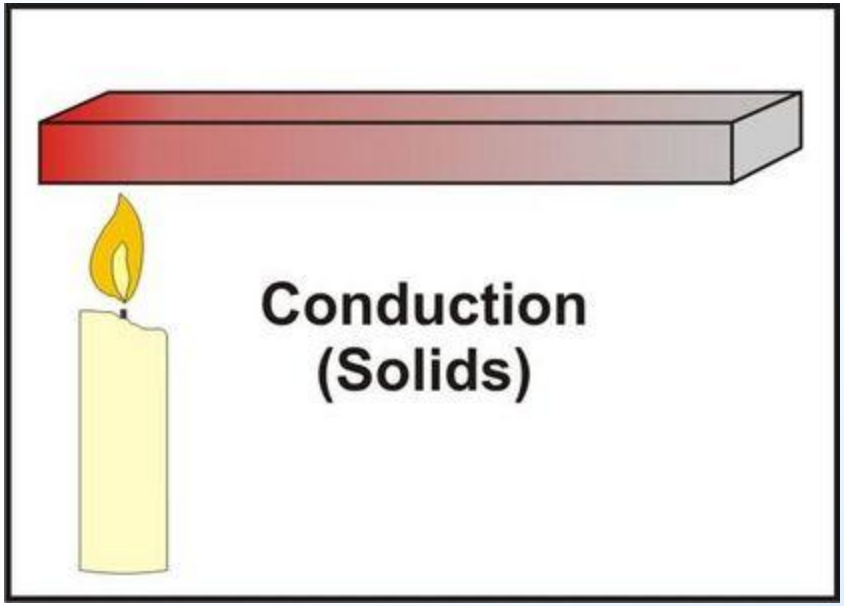
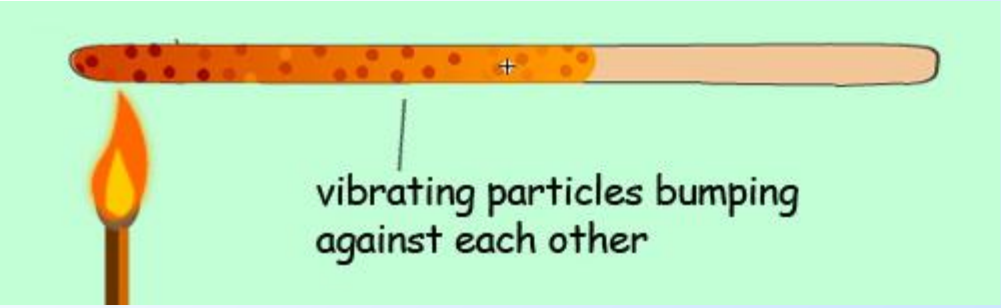
# HEAT TRANSFER

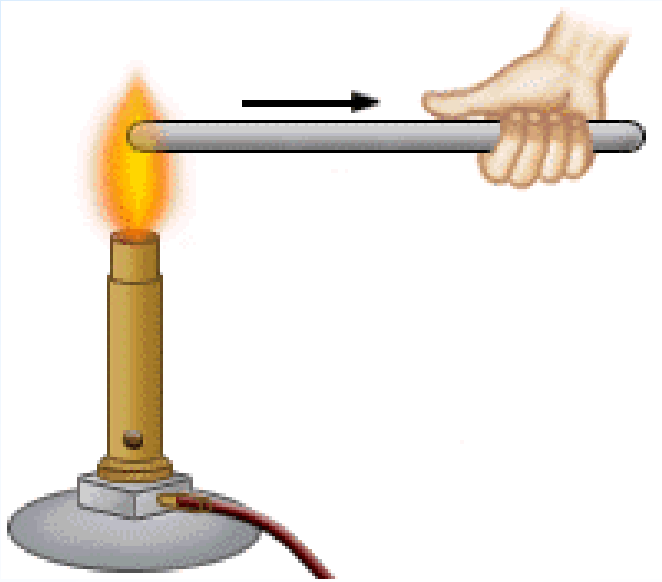


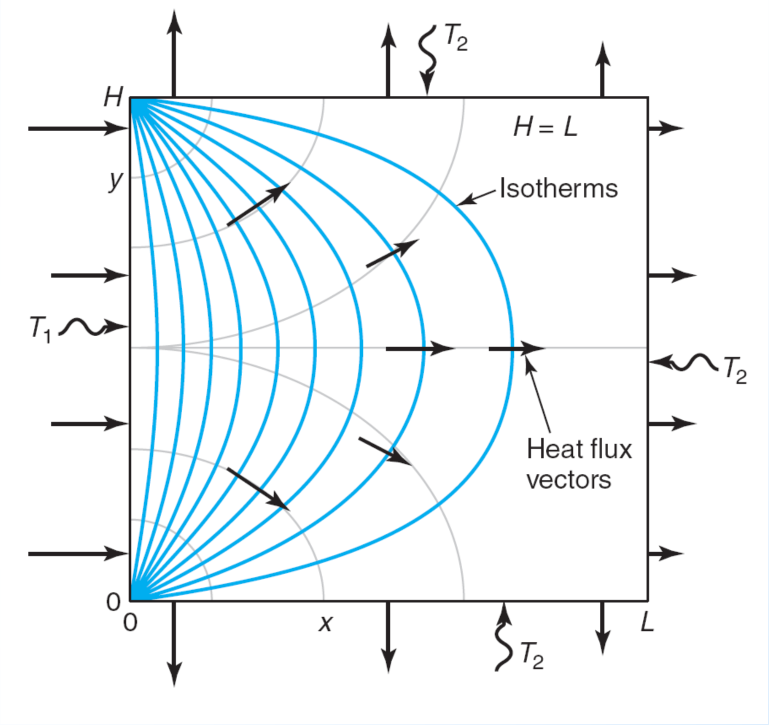

Conduction
Convection
Thermal radiation


Conduction
(Solids)


vibrating particles bumping against each other

# 2. CONDUCTION – Basic Equations






Heat transfer and temperature are closely related, but they are of different nature.

Temperature is a scalar quantity with only a magnitude.

Whereas heat transfer is a vector quantity with direction as well as a magnitude.

## Symbols and Units

T: Temperature, in ºC or K

t. Time, s

A: Area, $m^2$

k, or λ: Thermal conductivity, W/m.ºC or W/m.K

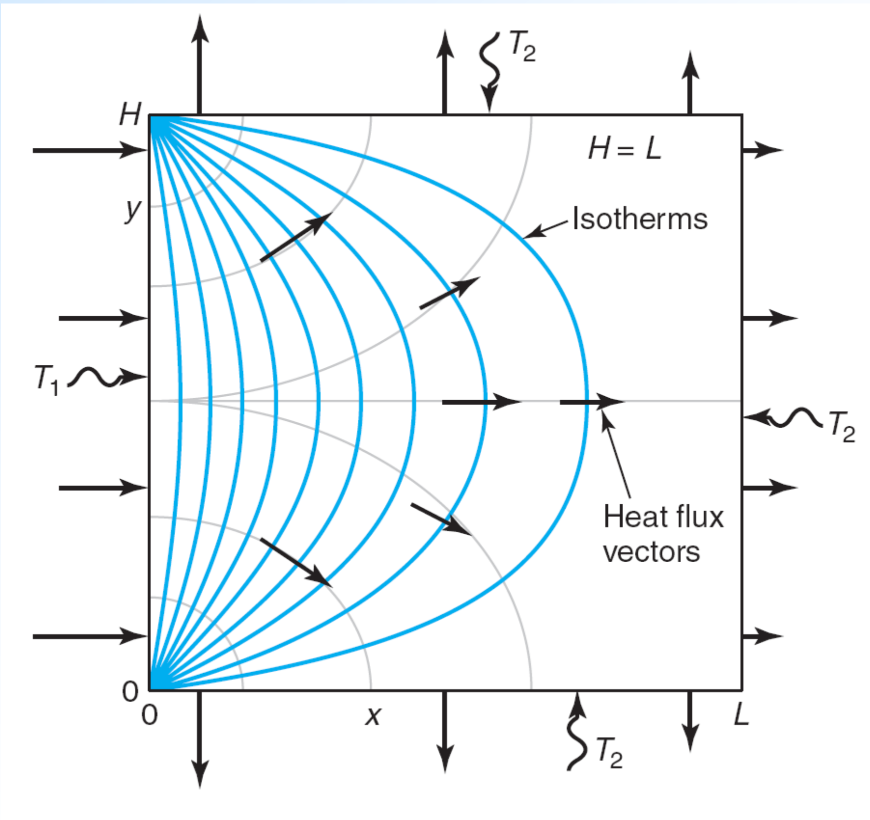
$\dot{Q}$ : Heat flow rate: J/s or W

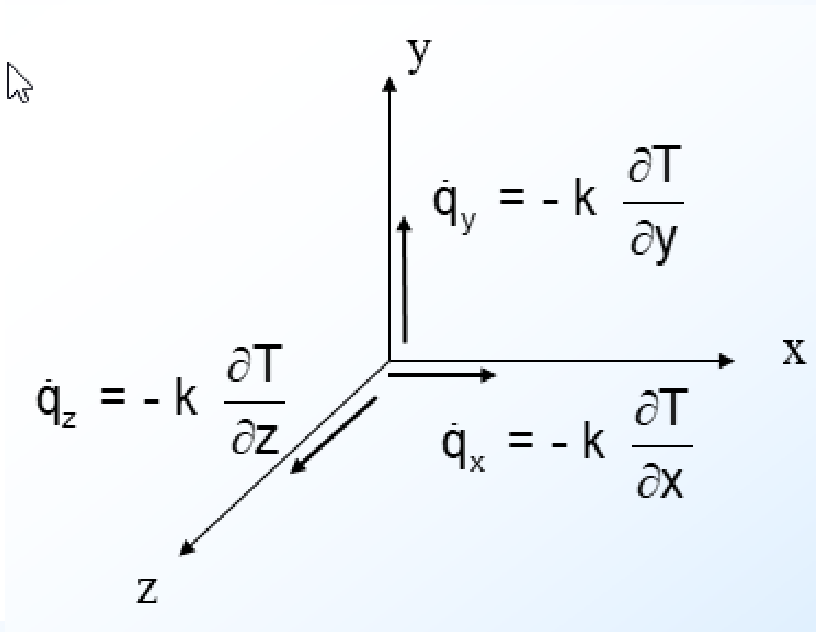
$\dot{q}$ , q''= Q/A: Heat flux, $W/m^2$

$\dot{e}_{gen}$, $\dot{g}_{gen}$, q''': Thermal energy generation rate, $W/m^3$

## 2.1. Heat Flux Components

In general, temperature varies in all directions, hence there is heat flow in those directions.



## Fourier's Law of heat conduction

$$\frac{\dot{Q}_{cond,x}}{A_x} = \dot{q}_{cond,x} = -k\frac{\partial T(t,x,y,z)}{\partial x}$$

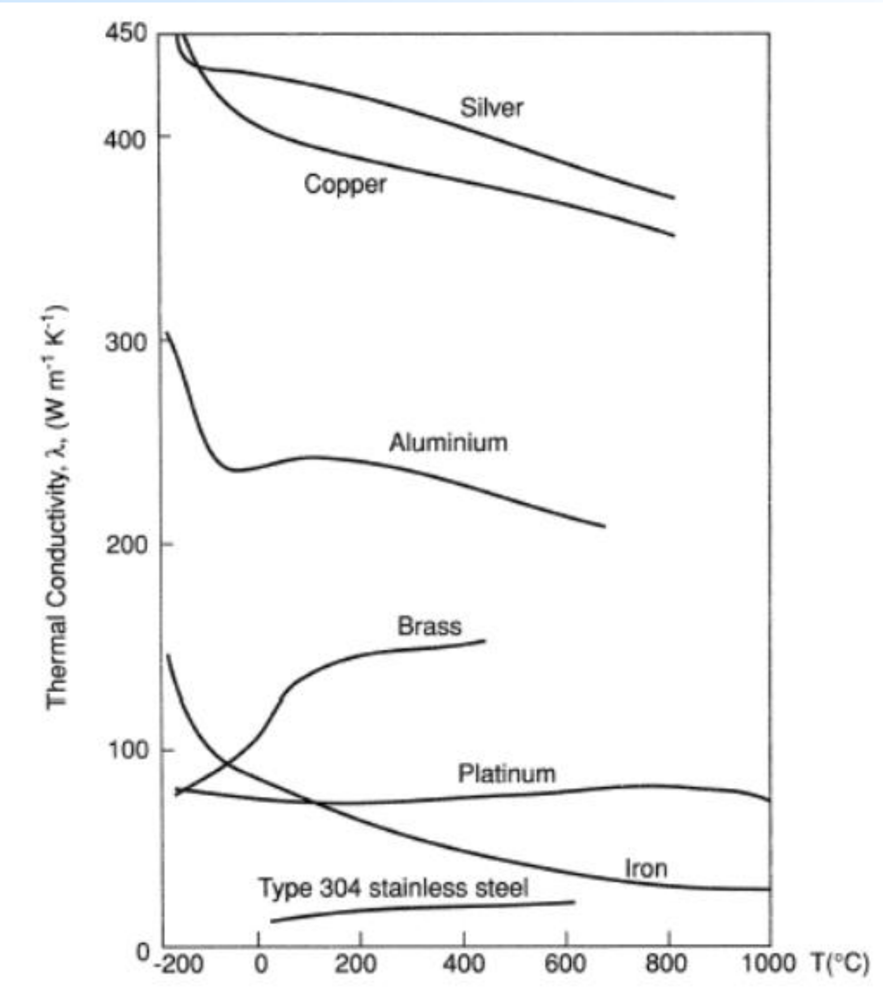
$$\frac{\dot{Q}_{cond,y}}{A_y} = \dot{q}_{cond,y} = -k\frac{\partial T(t,x,y,z)}{\partial y}$$

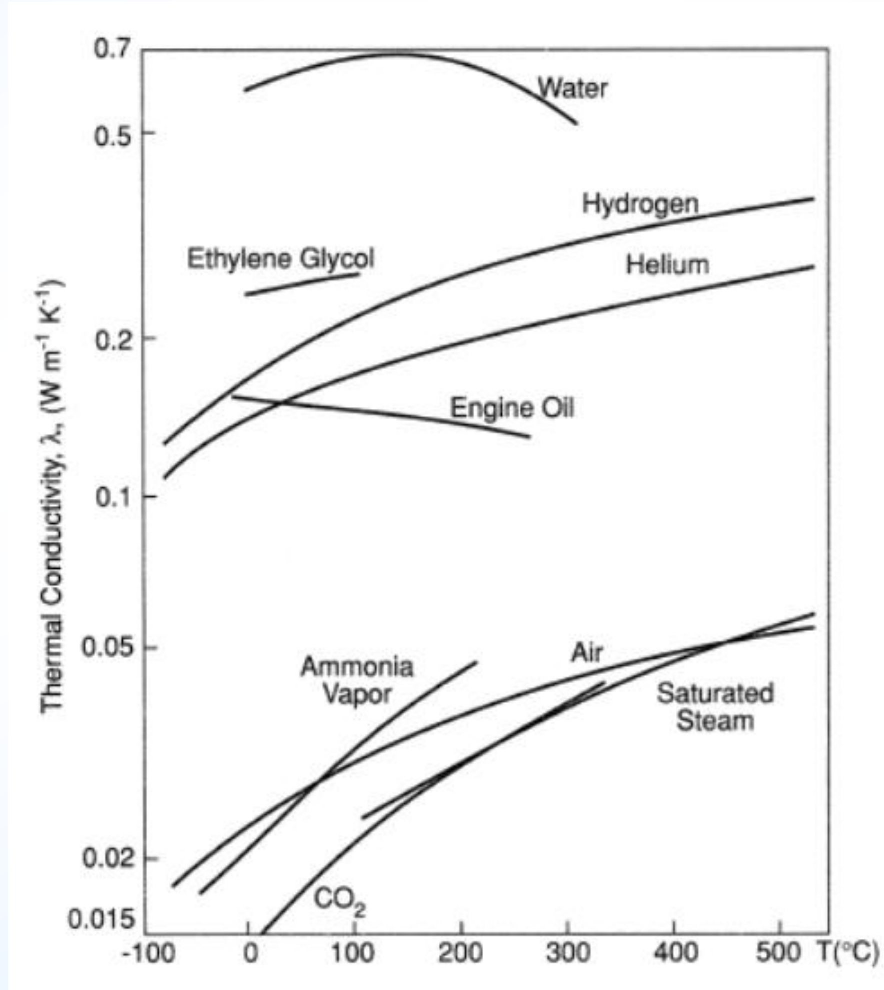
$$\frac{\dot{Q}_{cond,z}}{A_z} = \dot{q}_{cond,z} = -k\frac{\partial T(t,x,y,z)}{\partial z}$$

$$\dot{q}_{cond} = \dot{q}_x\,\vec{i} + \dot{q}_y\,\vec{j} + \dot{q}_z\,\vec{k}$$

Note that the thermal conductivity, k, at any given location does not vary at uniform temperature with the direction at that point for an isotropic medium, i.e., k is not a function of space variables. Exceptions: laminated sheets, crystals, wood (material with grains), etc.

The thermal conductivity, k or $\lambda$, may also vary with temperature.



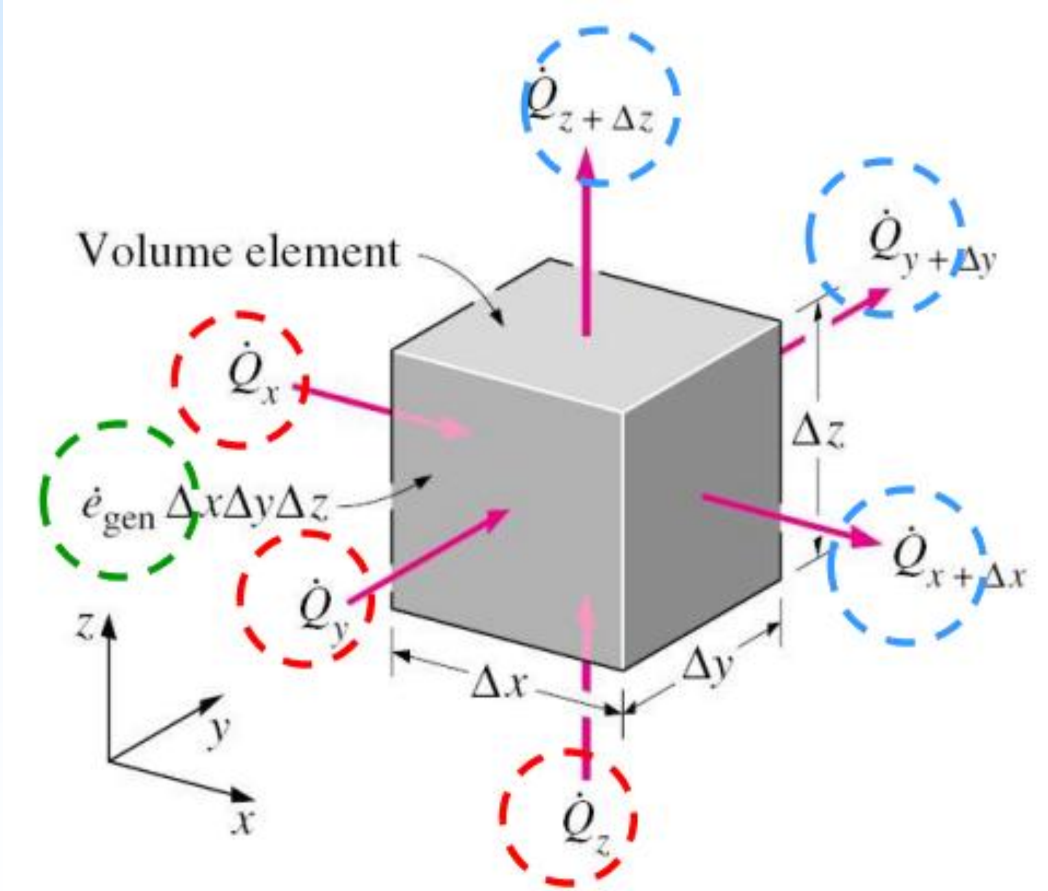



## 2.2. Differential Equation of Heat Conduction

The above equations imply that if the temperature distribution is known, then the rate of heat flow in all directions can be found.

The temperature distribution in a medium is determined from the solution of the differential equation of heat conduction subject to a set of appropriate boundary conditions.

Consider the following infinitesimaly small volume element ($\Delta x$ $\Delta y$ $\Delta z$) and write the energy balance:




[Net rate of heat flow entering by conduction into element Δx Δy Δz] + [Rate of energy generated in element Δx Δy Δz] = [Rate of increase of internal energy of element Δx Δy Δz]

Rate of heat conduction at *x*, *y*, and *z* − Rate of heat conduction at *x*+*Δx*, *y*+*Δy*, *and z*+*Δz* + Rate of heat generation inside the element = Rate of change of the energy content of the element

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + E_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right) + \dot{e}_{gen} = \rho\, c_p\, \frac{\partial T}{\partial t}$$

$\dot{e}_{gen}$ → q'''

## 2.3. Heat Conduction Equation in Other Coordinate System:

### Cartesian Coordinate System:

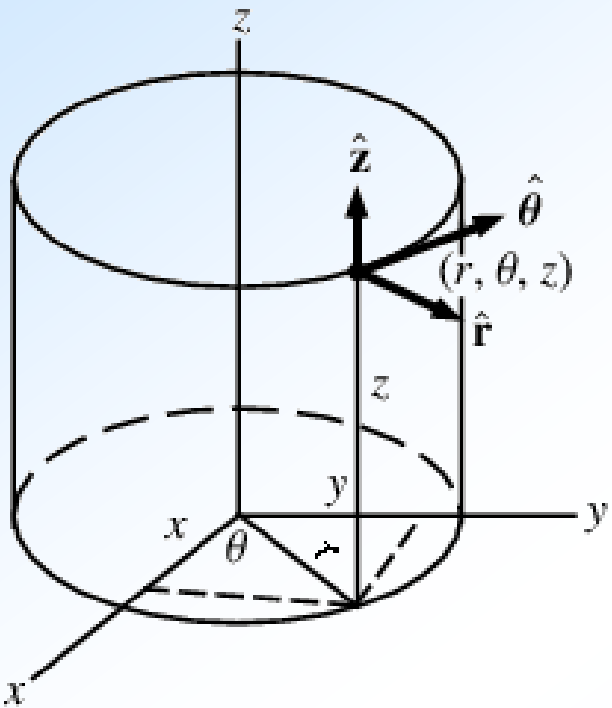
$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)+q'''=\rho\,c_p\,\frac{\partial T}{\partial t}$$

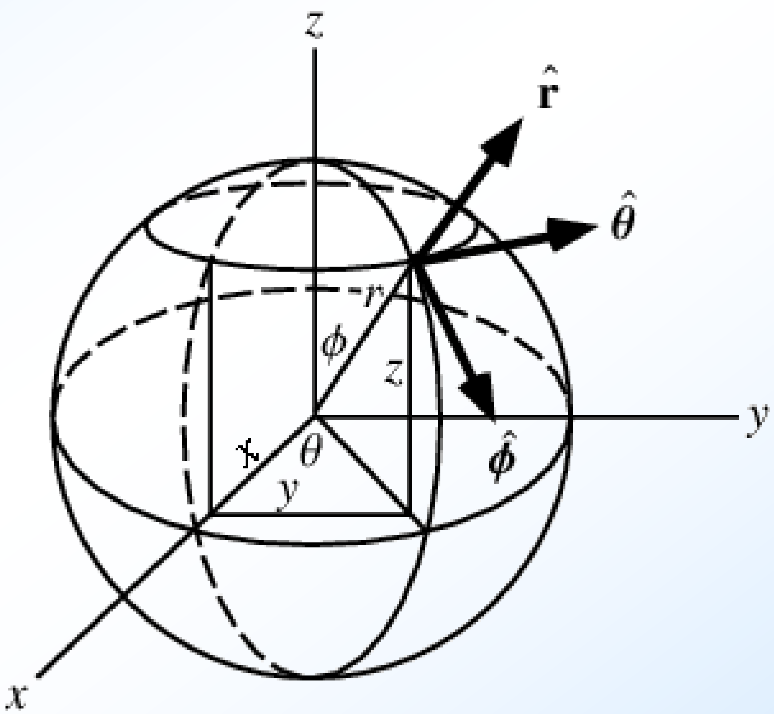
Heat conducted — Heat generated — Heat stored

### Cylindrical and Spherical Coordinate Systems:

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(k\,r\,\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\,\frac{\partial}{\partial \theta}\left(k\,\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)+q'''=\rho\,c_p\,\frac{\partial T}{\partial t}$$

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(k\,r^2\,\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2(\theta)}\,\frac{\partial}{\partial \phi}\left(k\,\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin(\theta)}\,\frac{\partial}{\partial \theta}\left(k\,sin(\theta)\,\frac{\partial T}{\partial \theta}\right)+q'''=\rho\,c_p\,\frac{\partial T}{\partial t}$$




Cylindrical Coordinate

Systems: (r, θ, z)

Spherical Coordinate

Systems: (ρ, θ, ϕ)

## General Methodology of Solution

- Solve the three-dimensional partial differential equation of heat conduction

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{k}\,\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}}\right)+\frac{\partial}{\partial \boldsymbol{y}}\left(\boldsymbol{k}\,\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{y}}\right)+\frac{\partial}{\partial \boldsymbol{z}}\left(\boldsymbol{k}\,\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{z}}\right)+\boldsymbol{q}'''=\rho\,\boldsymbol{c}_{\mathbf{p}}\,\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{t}}$$

  and find the temperature profile, T(x,y,z,t), in the solid.

- Use the initial condition (on time) and the given boundary conditions (two for each coordinates (x, y, and z for Cartezian system).
- Note that the thermal conductivity, k, may not be constant and can be a fuction of the space parameters (x, y, and z in Cartesiam system)

## General Methodology of Solution

- Use Fourier's law of heat condiction to find the heat flow rate in each direction ($Q_x$, $Q_y$, and $Q_z$ for Cartezian system).

$$Q_x = - k_x \; A_x \; \frac{\partial T}{\partial x}$$

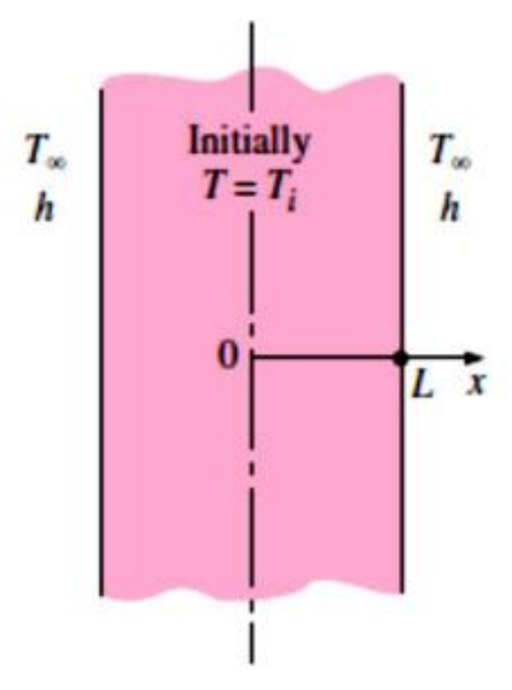
- Note that even A, the heat transfer area, can ve a function of the space coordinate, x

- Find $Q = Q_x + Q_y + Q_z$

## 2.4. Boundary Conditions:

- Specified temperature boundary condition
- Specified heat flux boundary condition
- Convection boundary condition
- Radiation boundary condition
- Interface boundary condition
- Generallized boundary conditions

## Specified temperature boundary condition



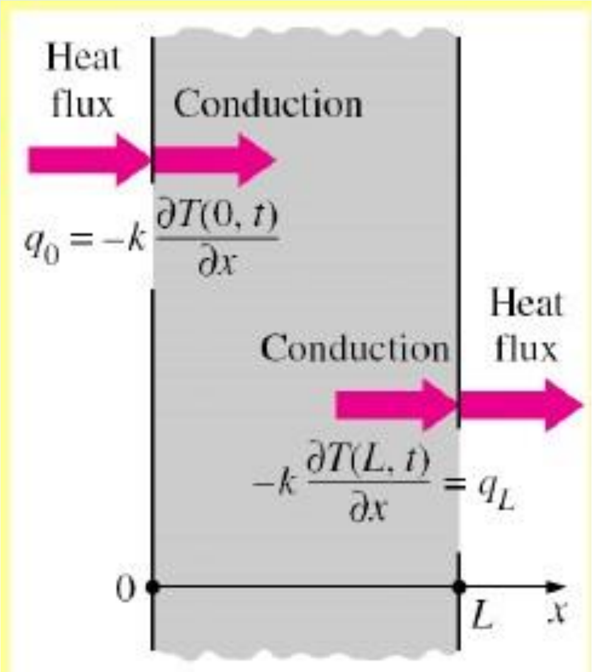

For one-dimensional heat transfer through a plane wall of thickness L, for example, the specified temperature boundary condition can be expressed as

$$\boldsymbol{T(0,t) = T_1}$$

$$\boldsymbol{T(L,t) = T_2}$$

The specified temperatures can be constant, which is the case for steady conduction, or may vary with time.

## Specified heat flux boundary condition




The heat flux in the positive x-direction anywhere in the medium, including the boundaries, can be expressed by Fourier's law of heat conduction as

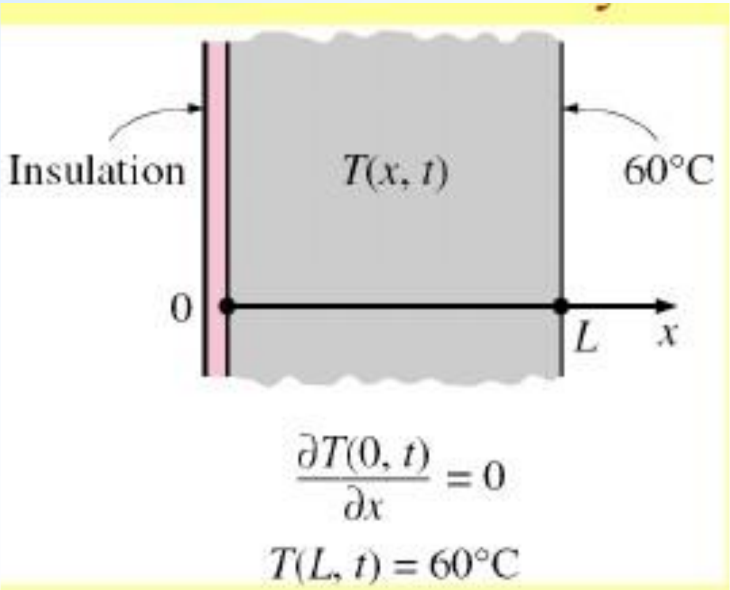
$$q'' = -k\,\frac{dT}{dx}$$

This is the heat flux in the positive x-direction

The sign of the specified heat flux is determined by inspection: positive if the heat flux is in the positive direction of the coordinate axis, and negative if it is in the opposite direction.

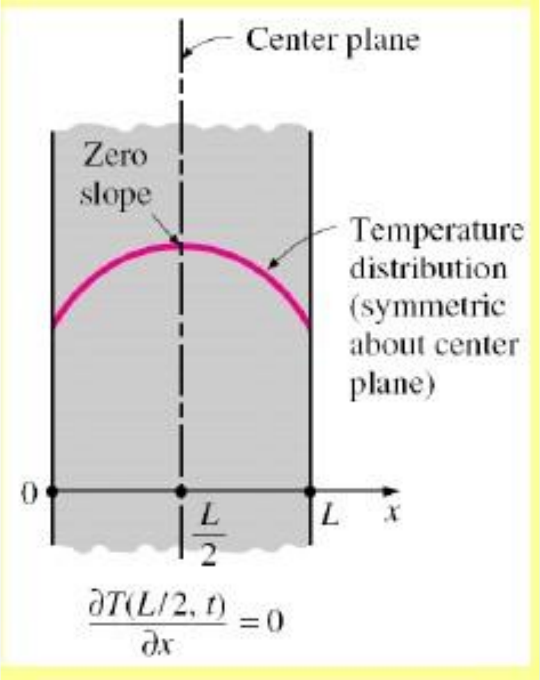
## Two special cases – Insulated boundary

Insulated boundary




$$k\frac{\partial T(0,t)}{\partial x} = 0 \quad or \quad \frac{\partial T(0,t)}{\partial x} = 0$$

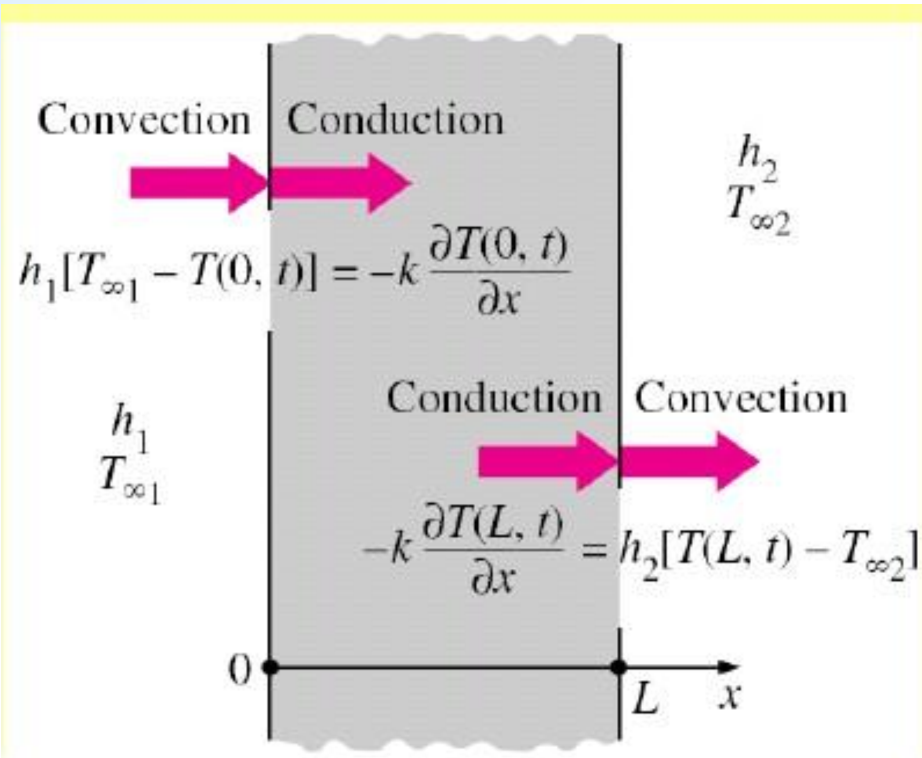
Thermal symmetry




$$\frac{\partial T(L/2,t)}{\partial x} = 0$$

## Convection boundary condition

(Heat conduction at the surface in a selected direction) = (Heat convection at the surface in the same direction)

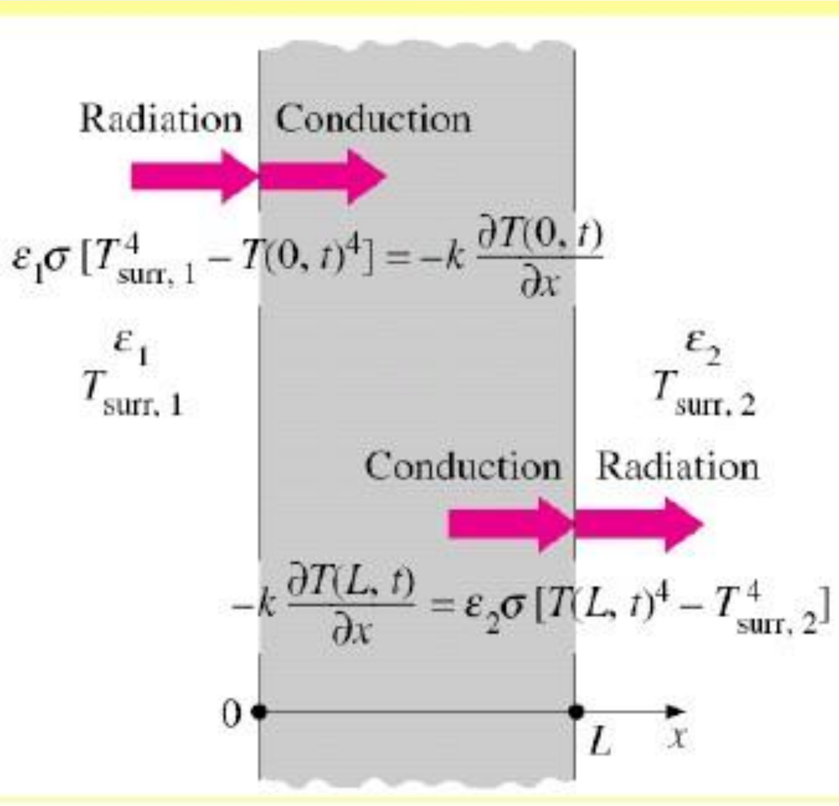



$$-k \frac{\partial T(0,t)}{\partial x} = h_1 [T_{\infty 1} - T(0,t)]$$

$$-k \frac{\partial T(L,t)}{\partial x} = h_2 [T(L,t) - T_{\infty 2}]$$

## Radiation boundary condition

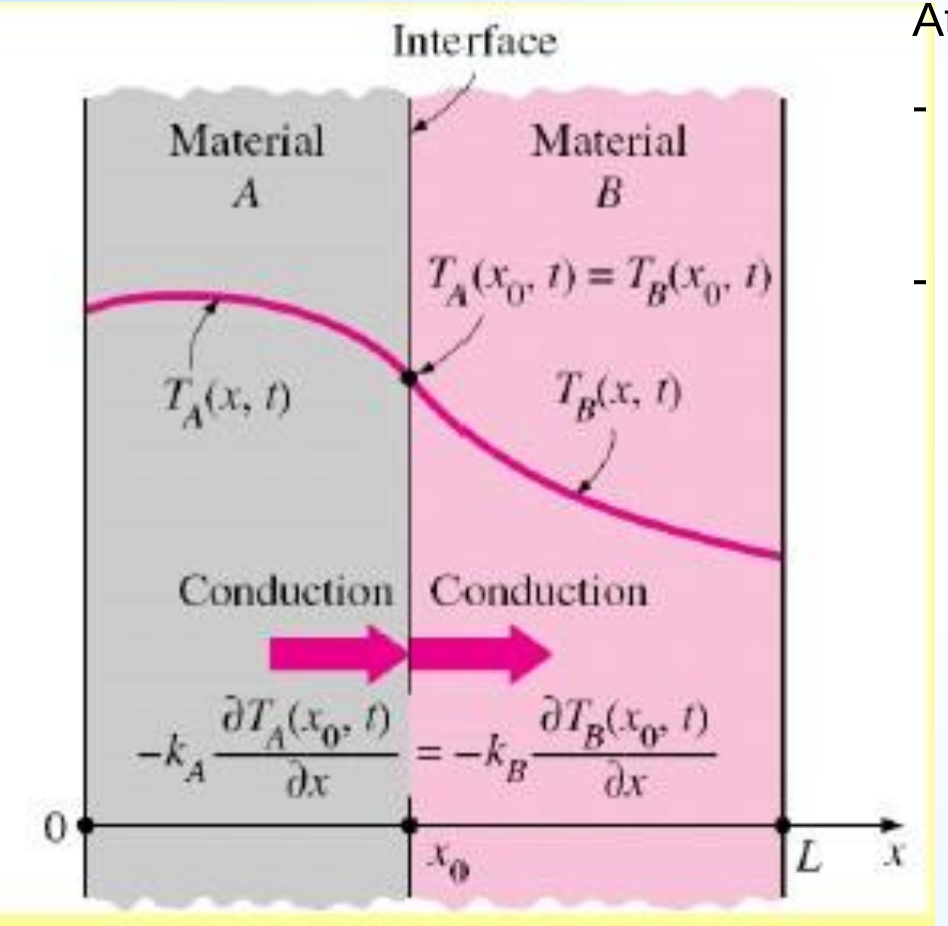
(Heat conduction at the surface in a selected direction) = (Radiation exchange at the surface in the same direction)




$$-k\,\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\,\sigma\,[T^4{}_{surr,1} - T(0,t)^4]$$

$$-k\,\frac{\partial T(L,t)}{\partial x} = \varepsilon_2\,\sigma\,[T(L,t)^4 - T^4{}_{surr,2}]$$

## Interface boundary conditions




At the interface, the requirements are:

- The same temperature at the area of contact,
- The heat flux on the two sides of an interface must be the same.

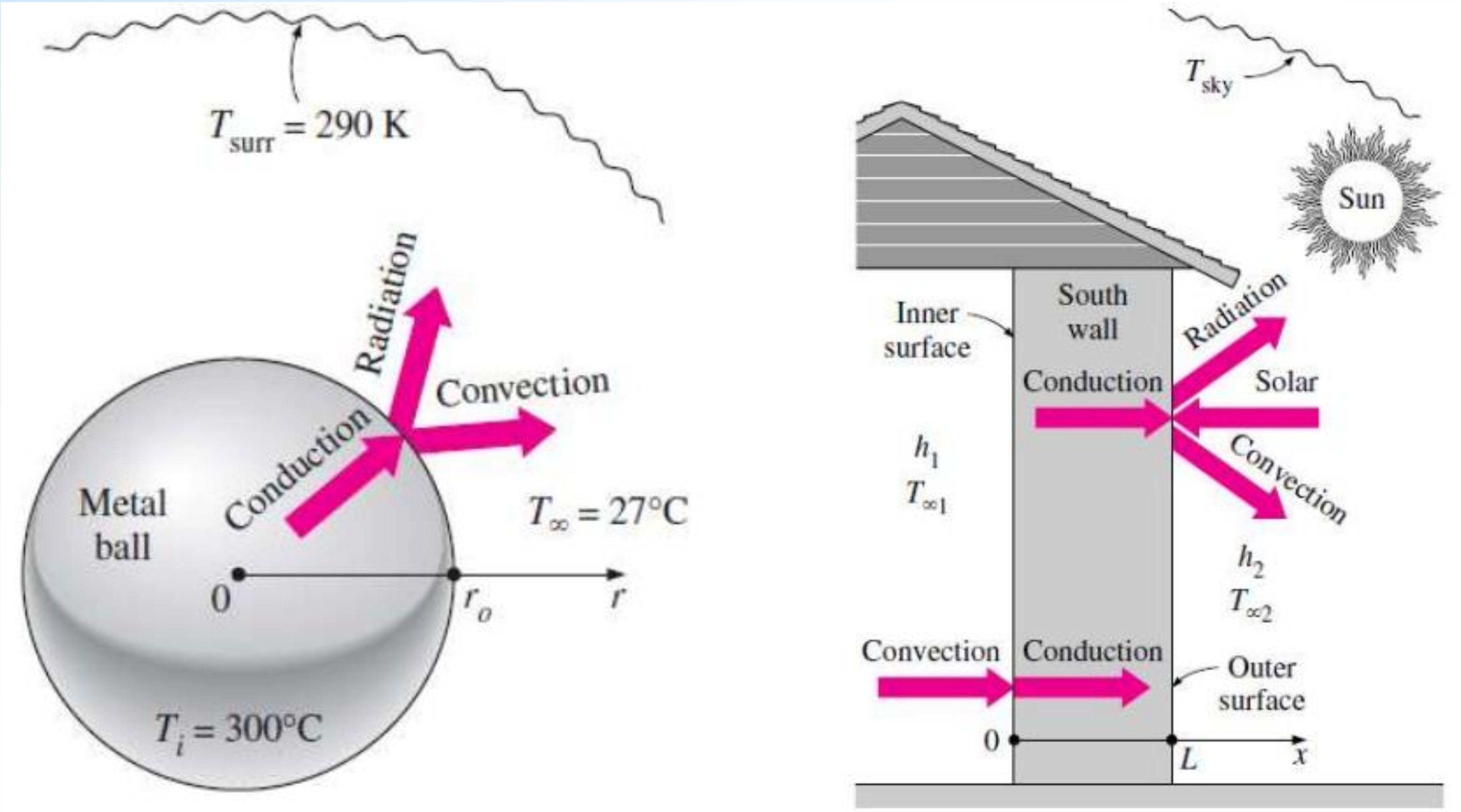
$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$

## Generalized boundary conditions

In general, a surface may involve convection, radiation, and specified heat flux, simultaneously. The boundary condition in such cases is again obtained from a surface energy balance, expressed as

$$\left(\begin{array}{c}\text{Heat transfer to the} \\ \text{surface in all modes}\end{array}\right) = \left(\begin{array}{c}\text{Heat transfer from the} \\ \text{surface in all modes}\end{array}\right)$$


T_surr = 290 K
Radiation
Convection
Conduction
Metal ball
T_∞ = 27°C
0
r_o
r
T_i = 300°C
T_sky
Sun
South wall
Inner surface
Radiation
Conduction
Solar
h_1
T_∞1
Convection
h_2
T_∞2
Convection
Conduction
Outer surface
0
L
x

## Simplified cases

Constant thermal conductivity

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{gen}}{k} = \overbrace{\frac{\rho\, c_p}{k}}^{1/\alpha} \frac{\partial T}{\partial t}$$

Constant thermal conductivity and steady state

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{gen}}{k} = 0$$

Poisson equation

Constant thermal conductivity, steady state, and no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

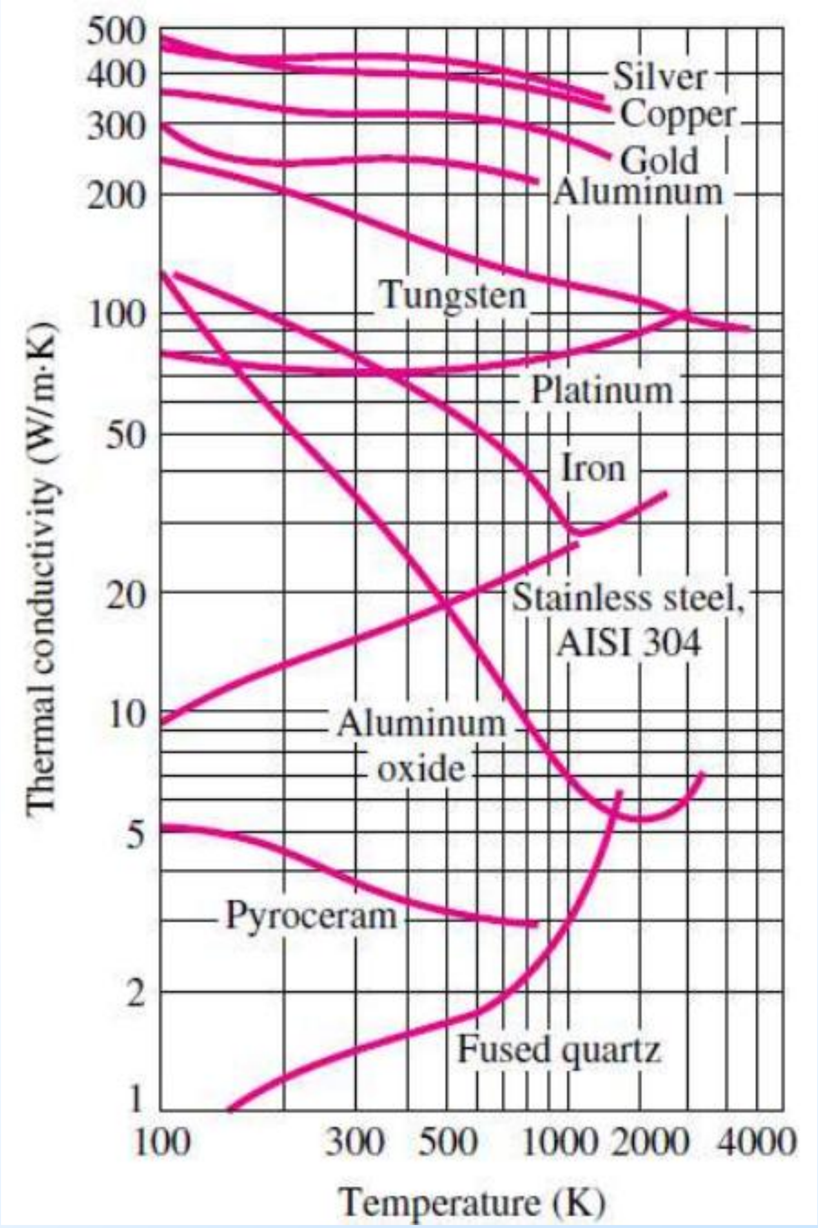
Laplace equation

**Variable thermal conductivity, k**

The thermal conductivity of a material, in general, varies with temperature.

An average value for the thermal conductivity is commonly used when the variation is mild.

This is also common practice for other temperature-dependent properties such as the density and specific heat.



## Heat generation in solids

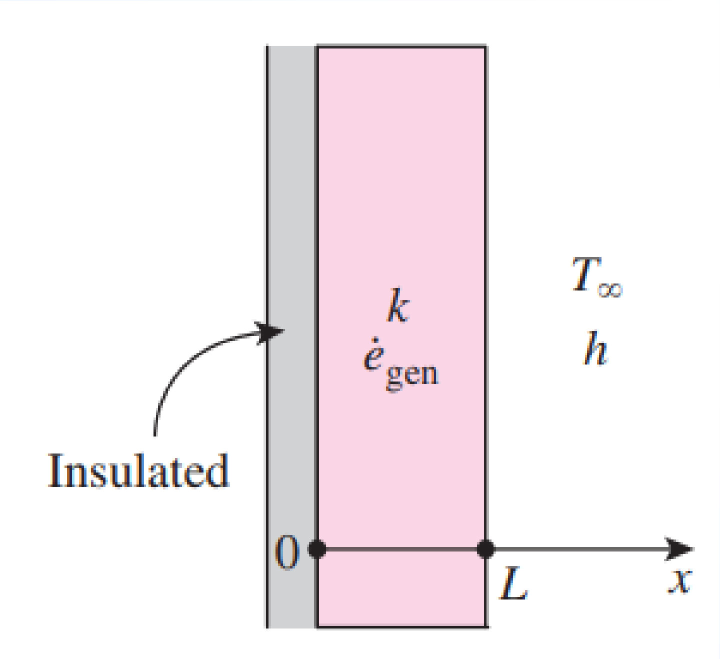
Resistance heating in wires $$\dot{e}_{gen} = \frac{\dot{E}_{gen,\ elecric}}{Vol} = \frac{I^2\ R_e}{\pi\ r^2\ L}$$

Exothermic chemical reactions in a solid

Nuclear reactions in fuel rods

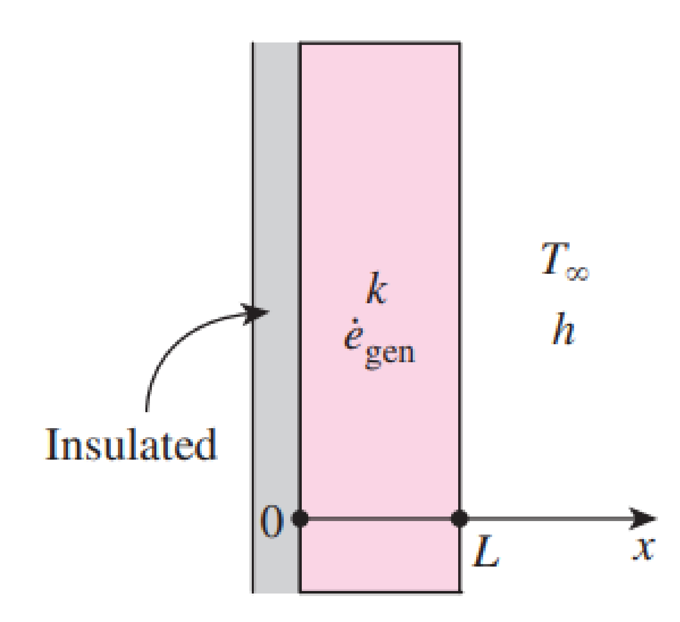
## 2.5. Non-dimensional heat conduction parameters

The number of variables in a heat conduction problem can be reduced by introducing non-dimensional parameters. Non-dimensional scaling provides a method for developing dimensionless groups that can provide physical insight into the importance of various terms in the system of governing equations.




Consider the following problem:

- A slab in the region $0 \le x \le L$ with constant thermal properties
- IC: at t = 0, T = T0 (uniform)
- BC's:: at x = 0 Insulated surface
  at x = L Convection
- There is heat generation $e_{gen}$




Differential equation: $$\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{\dot{e}_{gen}}{k} = \frac{\rho\, c_p}{k}\frac{\partial T}{\partial t}$$

Initial condition: $T(x,0) = T_0$ at $t = 0$ , $0 \le x \le L$

Boundary Conditions:

1) $$\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{at } x = 0 \ , \quad t > 0$$

2) $$k\,\frac{\partial T(L,t)}{\partial x} = h\left(T(L,t) - T_\infty\right) \quad \text{at } x = L \ , \quad t > 0$$

The differential equation can be non-dimensionalized by defining the following non-dimensional variables:

$$X = \frac{x}{L} \qquad \text{and} \qquad \theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

Differential equation: $\dfrac{\partial^2\theta}{\partial X^2} \dfrac{\dot{e}_{gen} L^2}{(T_0 - T_\infty) k} = \dfrac{\partial\theta}{\partial(\alpha t / L^2)}$ in $0 \leq X \leq 1$ for $t > 0$

Initial condition: $\theta = 1$ in $0 \leq X \leq 1$ for $t = 0$

Boundary Conditions:

1) $\dfrac{\partial\theta}{\partial X} = 0$ at $X = 0$ for $t > 0$

2) $\dfrac{\partial\theta}{\partial X} = \dfrac{h L}{k} \theta$ at $X = 1$ for $t > 0$

Define three non-dimensional parameters

Biot Number $Bi = \dfrac{h L}{k}$ Fourier Number $Fo = \dfrac{\alpha t}{L^2}$

Non-dimensional heat generation: $G = \dfrac{\dot{e}_{gen} L^2}{k (T_0 - T_\infty)}$

Bi and Fo are two important non-dimensional parameters frequently used in heat conduction problems

The Biot number, Bi, is the ratio of the thermal resistance for conduction inside a body to the resistance for convection at the surface of the body

Fourier Number, Fo, is a measure of the rate of heat conduction in comparison with the rate of heat storage in a given volume element.

$$\text{Fo} = \frac{\alpha\ t}{L^2} = \frac{k\ (1/L)\ L^2}{\rho\ c_p\ L^3\ /\ t} = \frac{\text{Rate of heat conduction across L in volume } L^3}{\text{Rate of heat storage in volume } L^3}$$