Exponential Growth & Decay Problem:

• Example: A certain radioactive material is known to decay at a rate proportional to the amount present. Half of this material have undergone disintegration in 2000 years:

(i) How much material will remain after 4000 years?

(ii) In how many years will only $\frac{1}{10}$ of the original material remain?

• Solution: $X(t)$ = Amount of material exist after "t" years

$$\frac{dX}{dt} \overset{\text{proportinality}}{\propto} t \iff \frac{dX}{dt} = -kt \quad (k>0)$$

• $(-)$ sign since material decreases.

• k = constant of proportionality

Let, $\underbrace{\boxed{X(0) = X_0}}_{\text{Initial amount of material at time } t=0}$ be the IC

$$\iff \underbrace{\frac{dX}{dt} = -kX}_{\text{seperable ODE}} \Rightarrow \frac{dX}{X} = -k\,dt \Rightarrow \ln|X| = -kt + C_1$$

$$\Rightarrow |X| = e^{-kt+C_1} = e^{-kt} \cdot \underbrace{e^{C_1}}_{= C}$$

$$\Rightarrow X = \mp C e^{-kt} \Rightarrow \boxed{X = C e^{-kt}} \rightarrow \text{GENERAL SOLN TO THE ODE}$$

Now insert $t=0$ and $X = X_0$ for the IC $X(0) = X_0$ in the general solution to find the constant C:

$$X_0 = C \cdot e^{-k \cdot 0} = C \cdot \underbrace{e^0}_{1} = C \Rightarrow \boxed{X_0 = C} \rightarrow \text{we found } C$$

$\Rightarrow$ The specific solution for the IC $X(0)=X_0$ for the ODE is:

$$X = C.e^{-kt} \Leftrightarrow \boxed{X(t) = X_0.e^{-kt}} \text{ ------ } (*)$$

→ solution of the IVP

We know that half of the material is left after 2000 years $\Rightarrow$ $(t=2000)$

$$X(2000) = \frac{\cancel{X_0}}{2} = \cancel{X_0}.e^{-k.2000}$$

half of the material in the beginning

$$\Rightarrow e^{-k.2000} = \frac{1}{2}$$

$$\Leftrightarrow \ln(e^{-k.2000}) = \ln(1/2)$$

$$\Leftrightarrow -k.2000 = \ln(1/2)$$

$$\Leftrightarrow k = -\frac{\ln(1/2)}{2000} = \frac{\ln((1/2)^{-1})}{2000}$$

$$\Leftrightarrow \boxed{k = \frac{\ln(2)}{2000}}$$

→ we found "k"

Now insert "k" in $(*)$ and we get:

$$\boxed{X(t) = X_0.e^{-\frac{\ln(2)}{2000}.t}}$$

→ Now all unknowns are found for the formula of decay

(i) After 4000 years we have $X(4000)$ amount of material:

$$X(4000) = X_0.e^{-\frac{\ln(2)}{2000}.4000} = X_0.e^{-2.\ln(2)} = X_0.e^{\ln(2^{-2})}$$

insert $(t=4000)$ in the formula

$$= X_0.e^{\ln(1/4)}$$

$$= X_0.\frac{1}{4}$$

amount left after 4000 years

$$= \boxed{X_0/4}$$

(ii) For the $\frac{1}{10}$ of the material remain we need to find the time (years) need to be passed by using the formula:

$$X(t) = \underbrace{\frac{\cancel{X_0}}{10}}_{\text{1/10 of the material in the beginning}} = \cancel{X_0} \cdot e^{-\frac{\ln(2)}{2000} \cdot t} \Rightarrow \frac{1}{10} = e^{-\frac{\ln(2) \cdot t}{2000}}$$

$$\Leftrightarrow \frac{1}{10} = e^{-\frac{t}{2000} \cdot \ln(2)}$$

$$\Leftrightarrow \frac{1}{10} = e^{\ln\left(2^{-\frac{t}{2000}}\right)}$$

$$\Leftrightarrow 1/10 = 2^{-t/2000}$$

$$\Leftrightarrow 1/10 = (1/2)^{t/2000}$$

Take the $\log_e = \ln$ of both sides here

$$\Rightarrow \ln(1/10) = \ln\left(1/2^{t/2000}\right) \Leftrightarrow \ln(1/10) = \frac{t}{2000} \cdot \ln(1/2)$$

$$\Rightarrow \frac{t}{2000} = \frac{\ln(1/10)}{\ln(1/2)}$$

$$\Rightarrow \boxed{t = 2000 \cdot \frac{\ln(1/10)}{\ln(1/2)}}$$

↳ this is the time that needs to be passed for 1/10 of the material to remain