MIDDLE EAST TECHNICAL UNIVERSITY

EE 501 Linear Systems Theory

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Field

Example

A set is a collection of objects, either concrete or abstract.

Definition

A field is a set F, together with two mappings of $F \times F \to F$, called addition and multiplication, written as $(a, b) \to a + b$ and $(a, b) \to ab$ respectively with the following properties:

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Field continued..

Example

Set of real numbers $\mathbb{R}$ with standard addition and multiplication.

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Example

Set of binary numbers with modulo 2 addition and multiplication.

$F = \{0, 1\}$

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Field continued..

Example

Let $F = \mathbb{R} \times \mathbb{R}$. Let us define $+$ and $\cdot$ as:
$x + y := (x_1 + y_1, x_2 + y_2)$,
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Note that this is nothing but complex number field $\mathbb{C}$. Then $0_F = (0, 0)$ and $1_F = (1, 0)$.

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Exercise

Let $F = (0, \infty) = \mathbb{R}_+$ (positive real numbers) Given $x + y := xy$, $x \cdot y := e^{\ln(x)\ln(y)}$, show that F satisfies the axioms of field. Find 1_F and 0_F.

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Question

Are polynomials a field? Are matrices a field?

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A linear space V is a set, whose elements are called vectors associated with a field F, whose elements are called scalars. Vectors can be added and they can be multiplied by scalars. These operations satisfy the following properties:

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Linear Spaces

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Example

Show that $0x = 0$

Example

Set of all vectors $(a_1, a_2, \ldots, a_n)$ with $a_i \in F$. Addition, multiplication are defined componentwise. This space is denoted as F^n. Let $x, y \in F^n$
$x = (a_1, a_2, \ldots, a_n)$, $y = (b_1, b_2, \ldots, b_n)$
Addition: $x + y := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
Multiplication: $cx := (ca_1, ca_2, \ldots, ca_n)$

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Addition: $x + y := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
Multiplication: $cx := (ca_1, ca_2, \ldots, ca_n)$
Most common examples are $\mathbb{R}^n$ and $\mathbb{C}^n$.

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Set of all polynomials with degree less than n with coefficients in F. Note that this linear space is a subset of the previous one for $F = \mathbb{R}$.

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Remark

Subset has to be closed under addition and scalar multiplication. All other properties are inherited from the original linear space.

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linear space $V = \mathbb{R}^2$;
subspace $W = [a\ 0]^T : a \in \mathbb{R}$

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Example

linear space V = set of all real valued functions $t \to f(t)$;
subspace W_1 = set of all continuous functions,
subspace W_2 = set of all functions periodic with π.

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Example

Show that $Y + Z$ is a linear subspace of X if Y and Z are

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If Y and Z are subspaces of linear space X, is their union $Y \cup Z$ a subspace?

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- $a(v, w) := (av, aw)$ (scalar multiplication)

Definition

A linear combination of n vectors $x_1, x_2, , \ldots, x_n$ of a linear space C is a vector of the form $a_1 x_1, +a_2 x_2 +, \ldots, +a_n x_n,$ where a_i's are scalars in F.

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Vectors $x_1, x_2, , \ldots, x_n$ in X are said to be linearly independent iff $a_1x_1, +a_2x_2+, \ldots, +a_nx_n, = 0$ implies $a_i = 0, \forall i$. Otherwise, they are linearly dependent.

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Example

Consider the linear space of polynomials with degree $n \leq 2$. Let subset S=$\{P_1, P_2, P_3\}$ be such that $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$, $\quad \forall t$
Is this set linearly independent?

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Let V be a linear space and (finite) set of vectors S=$\{x_1, \ldots, x_n\}$ be a subset of V. S is said to be a basis for V iff

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Definition

A (finite dimensional) linear space V has many bases. All these bases must have the same number of vectors. That number is called the dimension of V.

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If $(x_1, x_2, \ldots, x_n)$ is an ordered basis of V and $y \in V$, then there is a unique n-tuple of scalars $(a_1, a_2, \ldots, a_n)$ such that $y = \sum_{i=1}^{n} a_i x_i$.

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Example

With respect to some ordered basis $B_1 = (x_1, x_2)$ of $\mathbb{R}^2$, let the vectors y_1, y_2, y_3 be presented by $[y_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $[y_2]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[y_3]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. That is, $y_1 = 1x_1 + 1x_2$, $y_2 = 1x_1 + 0x_2$, $y_3 = 2x_1 + 3x_2$. Let our new basis be $B_2 = (y_1, y_2)$. Express y_3 w.r.t. this new basis.

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Remark: For a given ordered basis, the representation of a vector is unique.

Normed linear spaces

Definition

Consider a linear space V over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. Let there be a function $x \to \|x\|$ that assigns to each $x \in V$, a nonnegative real number $\|x\| \in \mathbb{R} > 0$. Such function is called a **norm** if it satisfies the following properties.

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(P3) $\|x\| = 0 \Leftrightarrow x = 0$

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(P2) $\|\alpha x\| = |\alpha| \, \|x\| \ \forall x \in V$ and $\alpha \in F$

(P3) $\|x\| = 0 \Leftrightarrow x = 0$

The expression "‘$\|x\|$"’ is read "‘the norm of x"’ and the function $\|.\|$ is said to be a norm on V.

Normed linear spaces

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The triplex $(V, F, \|.\|)$ is called a **normed space**.

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Now that we have a proper tool for measuring distance (norm), we can begin studying the "geometry" of the space (parallelism, orthogonality, area, volume, shape in general).

Example

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iii) $\|x\|_\infty :=$ max$(|\alpha_1|, |\alpha_2|)$. Is $\|.\|_\infty$ a norm?

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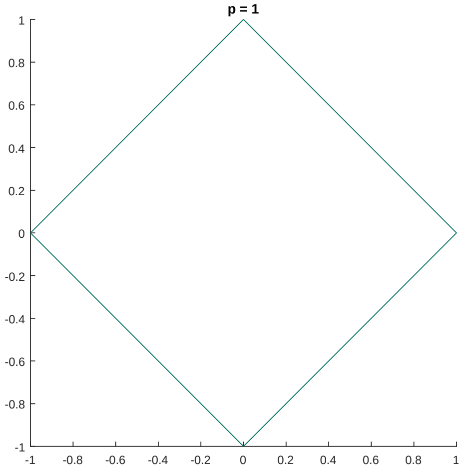
Note that $\lim_{p \longrightarrow \infty} \|x\|_p = \|x\|_\infty$.

Normed linear spaces

p-norm:



Normed linear spaces

Examples: On lecture notes...

Matrix Norms

Example

Let $V = \mathbb{R}^{n\times m}$, and $A = [a_{ij}]$,

$$\|A\| = \max_{i,j} |a_{ij}| \text{ is a norm.}$$

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Exercise

Show that this is a norm.

Matrix Norms

Definition

$A : \mathbb{R}^n \to \mathbb{R}^m$ be an $m \times n$ matrix. Let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ denote the norms (vector norms) in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The **induced norm** of a matrix is defined as

$$\|A\| := \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_{\mathbb{R}^m}}{\|x\|_{\mathbb{R}^n}}.$$

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Remark:

The induced matrix norm is defined in terms of vector norms. An equivalent definition is:

$$\|A\| := \max_{\|x\|=1} \|Ax\|.$$

Remark:

$$
\begin{aligned}
\|Ax\| &= \frac{\|Ax\|}{\|x\|}\,\|x\| \;(\text{ suppose } \;\|x\| \neq 0) \\
&\leq \left(\max_y \frac{\|Ay\|}{\|y\|}\right)\|x\| \\
&= \|A\|\,\|x\| \Rightarrow \|Ax\| \leq \|A\|\,\|x\|
\end{aligned}
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Matrix Norms

Remark:

$$\|Ax\| = \frac{\|Ax\|}{\|x\|}\|x\| \, (\text{ suppose } \; \|x\| \neq 0)$$
$$\leq \left(\max_y \frac{\|Ay\|}{\|y\|}\right)\|x\|$$
$$= \|A\|\,\|x\| \Rightarrow \|Ax\| \leq \|A\|\,\|x\|$$

Furthermore, there exists a vector x^* such that $\|Ax^*\| = \|A\|\,\|x^*\|$ which may not be unique.

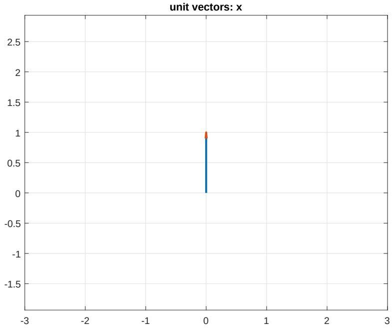
Example

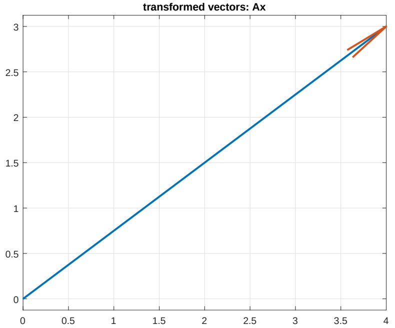
Choose $\|.\|_2$ in $\mathbb{R}^n$ and $\mathbb{R}^m$,

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \sqrt{(Ax)^T Ax} = \max_{\|x\|=1} \sqrt{x^T A^T Ax}$$

Induced Norm Example

Consider the unit vectors in $\mathbb{R}^2$ with $\|x\|_2 = 1$, and matrix $A = \begin{bmatrix} 1 & 4 \\ -6 & 3 \end{bmatrix}$,

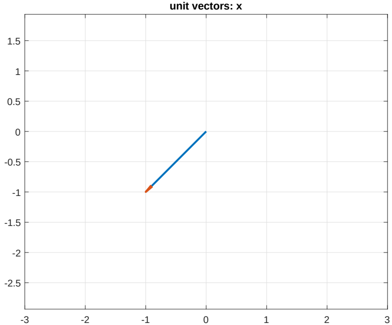


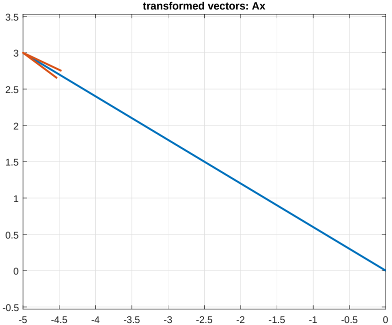


Note that, $\max \|Ax\|_2 = 6.80$, and $\sqrt{\lambda_{max}(A^T A)} = 6.80$

Induced Norm Example

Consider the unit vectors in $\mathbb{R}^2$ with $\|x\|_\infty = 1$, and matrix $A = \begin{bmatrix} 1 & 4 \\ -6 & 3 \end{bmatrix}$,

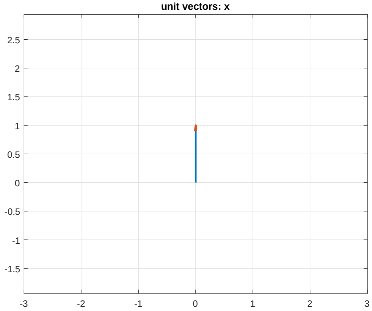


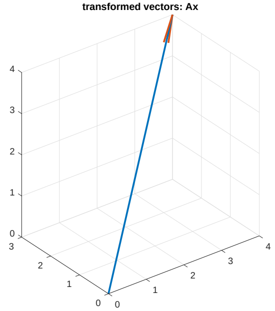


Note that, $\max \|Ax\|_\infty = 9.00$, and maximum absolute row sum of A $= 9.00$

Induced Norm Example

Consider the unit vectors in $\mathbb{R}^2$ with $\|x\|_2 = 1$, and matrix $A = \begin{bmatrix} 1 & 4 \\ -6 & 3 \\ 7 & 4 \end{bmatrix}$,





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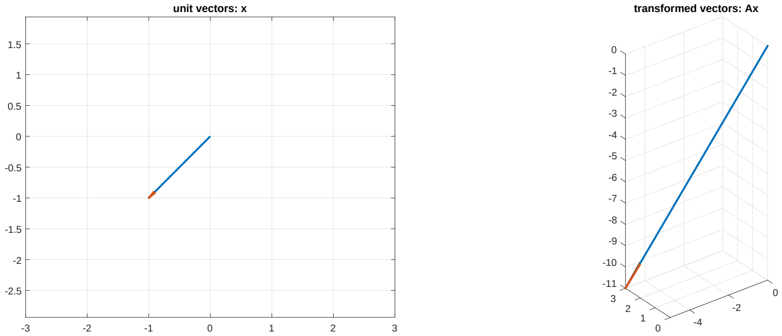

Note that, $\max \|Ax\|_\infty = 11.00$, and maximum absolute row sum of A $= 11.00$

Definition

Let $(V, F, \|.\|)$ be a normed space. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of vectors in V. $v_n \in V \;\; n = 1, 2, \ldots$. The sequence is said to be **convergent** to the limit $\bar{v} \in V$ iff $\|v_n - \bar{v}\| \to 0$ as $n \to \infty$

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Remark:

A sequence that is not convergent is called **divergent**.

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Given $V = \mathbb{R}$ and $\|v\| = |v|$, consider the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty}$.
Is the sequence convergent to $\bar{v} = 0$.

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Example

Consider the sequence $\{(-1)^n\}_{n=1}^{\infty}$

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In most engineering applications, we are interested in the convergence of an iterative algorithm.

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The given definition of the convergence requires the limiting element $\bar{v} \in V$, an element we may not know, to verify convergence.

There are ways to exclude "$\bar{v} - dependence$".

Cauchy sequence

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Let $(V, F, \|.\|)$ be a normed space. A sequence $\{v_n\}_{n=1}^{\infty}$ in V is said to be a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N$(depending on ϵ) such that $\|v_n - v_m\| < \epsilon$ for all $n, m > N$.

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Every convergent sequence is a Cauchy sequence. The converse, in general, is not true.

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Remark:

Every convergent sequence is a Cauchy sequence. The converse, in general, is not true.

Example

Consider the normed space $(\mathbb{Q}, \mathbb{Q}, |.|)$, (i.e., set of rational numbers over the field rational numbers with norm being the absolute value). Is the sequence $\left\{1 + \sum_{i=1}^{n} \frac{1}{i!}\right\}_{n=1}^{\infty}$ convergent?

Banach Space

Definition

A normed space is said to be **complete** if every Cauchy sequence is convergent. A complete normed space is called a **Banach Space**.

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Example

"'A normed space that is not complete"'

Let $V = \{f | f : [-1, 1] \to \mathbb{R},$ f is continuous and $\int_{-1}^{1} |f(t)| dt < \infty\}$.

Define $\|f\|_1 := \int_{-1}^{1} |f(t)| dt$.

Now consider the sequence $\{f_n\}_{n=1}^{\infty}$ defined as follows:

Inner Product Space

An inner product space is a linear space with an additional structure called inner product.

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Definition

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   - 2a) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ (linearity in the first argument)
   - 2b) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ (linearity in the first argument)

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3) $\langle x, x \rangle \geq 0$ with equality only for $x = 0$ (positive defineteness)

Inner Product Space

Notice that

$$\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$$

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$$\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$$

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Example

$\ldots$

Example

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Example

...

Theorem

Cauchy-Schwarz inequality:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Summary

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- Sets

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Very general concept. We can perform:

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- Take unions, intersections, complements, set subtraction

Summary

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- define **linear transformations** between vector spaces.

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Normed spaces have a major shortcoming. The direction cannot be characterized. The direction, or rather **relative direction**, can be studied by the help of a tool we call the **inner product**.

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Let u an v be two unit vectors in $\mathbb{R}^n$

- $\langle u, v \rangle = 0$ if they are orthogonal
- $\langle u, v \rangle$ is maximum when u and v point in the same direction
- $\langle u, v \rangle$ is minimum when u and v point in the opposite direction

Linear Transformation

Definition

Let V and W be linear spaces over the same field F. A linear transformation T is a mapping $T : V \to W$ satisfying
$T(a_1x_1+a_2x_2) = a_1T(x_1)+a_2T(x_2) \qquad \forall a_1, a_2 \in F \quad and \quad \forall x_1, x_2 \in V$

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- $V = W$ polynomials of degree less than n in S; $T = \frac{d}{ds}$
- $V = W =$\{continuous functions of type $f : [0,1] \to \mathbb{R}$\};
  $T_f = \int_0^1 f(s)ds$

Range and Null Spaces

Definition

Given linear transformation $T : V \to W$, the **null space** of T is the set of all $x \in V$ satisfying $Tx = 0_w$. That is,
$\mathcal{N}(T) := \{x \in V : Tx = 0\}$

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Definition

Given linear transformation $T : V \to W$, the **range space** of T the set of all $w \in W$ satisfying $Tv = W$. That is
$\mathcal{R}(T) := \{w \in W : w = Tv \quad \text{for some} \quad v \in V\}$

Range and Null Spaces

Remark

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Theorem

Let $T : V \to W$ be a linear transformation. Then mapping T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$.