SOLUTION KEY

| Section: 71 |

Name & Surname: ____________________

Math 120 Spring 2017-2018
Quiz no.: 02
Date: 16.03.18
Time Limit: ~15 Minutes

ID Number: ____________________

Grade:

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1. Let $f(x) = \dfrac{e^x - 1}{x}$, $x \neq 0$ and $f(0) = 1$. Find Maclaurin series representation for the function and compute its radius of convergence. Find $f^{(100)}(0)$.

The Maclaurin series representation for e^x is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

Since this representation is valid for all $x \in \mathbb{R}$, we can substitute into $f(x)$:

$$f(x) = \frac{\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - 1}{x} = \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \underset{n \leftrightarrow n+1}{=} \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

To find Radius of convergence, apply Ratio Test: Let $a_n = \frac{x^n}{(n+1)!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{x^n} \right| = 0$$ so this series is convergent for all $x \in \mathbb{R}$,

the radius of convergence $R = \infty$.

For an analytic function $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} a_n x^n$ is the Maclaurin Series representation of this function.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \sum_{n=0}^{\infty} a_n \cdot x^n \quad \text{where} \quad a_n = \frac{1}{(n+1)!} = \frac{f^{(n)}(0)}{n!} \Rightarrow f^{(n)}(0) = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$f^{(100)}(0) = \frac{1}{100+1} = \frac{1}{101}.$$