SOLUTION KEY

Section:34

Name & Surname: ______________

Math 120 Spring 2017-2018
Quiz no.: 01
Date: 15.03.18
Time Limit: ~15 Minutes

ID Number: ______________

Grade:

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1. Let $f(x) = \int_0^x \sin t^2 dt$. Find Maclaurin series representation for the function and compute its radius of convergence. Find $f^{(100)}(0)$.

The Maclaurin series of $\sin x$ is $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all $x \in \mathbb{R}$, then we can find $\sin t^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{4n+2}$, for all $t \in \mathbb{R}$ by replacing x by t^2.

Substitute into $f(x)$;

$$f(x) = \int_0^x \sin t^2 dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{4n+2} \right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{t^{4n+3}}{(4n+3)} \Bigg|_0^x$$ by Term-by-Term integration

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)} \cdot x^{4n+3}$$

Since Maclaurin Representation of $\sin t^2$ is valid for all t, the radius of convergence is ∞. Moreover, integral of this series have the same radius convergence; hence, for $f(x)$ the radius of convergence $R = \infty$.

[OR: Apply Ratio test $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(2n+3)!(4n+7)} \frac{x^{4n+7}}{} \frac{(2n+1)!\,(4n+3)}{(-1)^n\, x^{4n+3}} \right| = 0$, so the series is convergent for all x, radius of convergence $R = \infty$]

For an analytic function; $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ is the Maclaurin representation then we have by using this representation;

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (4n+3)!}{(2n+1)!\,(4n+3)} \frac{x^{4n+3}}{(4n+3)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (4n+2)!}{(2n+1)!} \cdot \frac{x^{4n+3}}{(4n+3)!} = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$$ where $k = 4n+3$

$$a_k = \begin{cases} \frac{(-1)^n (4n+2)!}{(2n+1)!} & \text{if } k = 4n+3 \\ 0 & \text{otherwise} \end{cases}$$ and $a_k = f^{(k)}(0)$.

$f^{(100)}(0) = a_{100} = 0$ since $100 \neq 4n+3$ for any $n \in \mathbb{N}$.