26 Kasım 2021 Cuma 13:41

Q1| Determine whether $f(x)$ is continuous or not where $f(x) = \cos x^2$

The fact: If $f(x)$ and $g(x)$ are continuous then $(f \circ g)(x)$ and $(g \circ f)(x)$ are continuous.

In our case $f(x) = x^2$ and $g(x) = \cos x$. These two functions are continuous, so $\cos x^2$ is continuous everywhere.

Q2| Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} x \lfloor \frac{1}{x} \rfloor & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Find all the points at which $f(x)$ is discontinuous.

If $x \in (1, \infty)$ then $0 \leq \frac{1}{x} < 1 \Rightarrow \lfloor \frac{1}{x} \rfloor = 0 \Rightarrow f(x) = 0$

If $x \in [-\infty, -1]$ then $\frac{1}{x} \geq -1 \Rightarrow \lfloor \frac{1}{x} \rfloor = -1 \Rightarrow f(x) = -x$

If $x \in (\frac{1}{n+1}, \frac{1}{n}]$ then $\frac{1}{n+1} < x \leq \frac{1}{n} \Rightarrow \lfloor \frac{1}{x} \rfloor = n$

$$\Rightarrow \frac{1}{x} \geq n$$

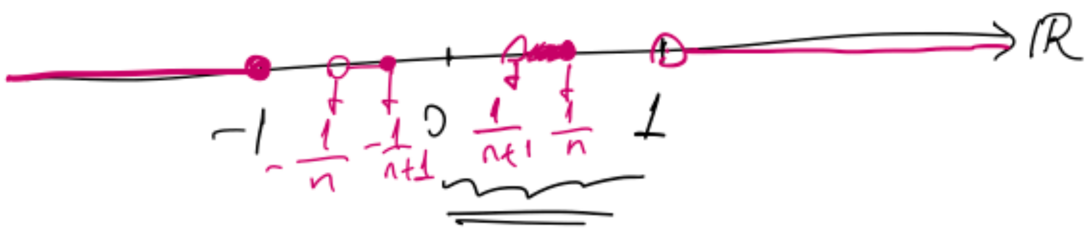
$$\Rightarrow f(x) = nx$$

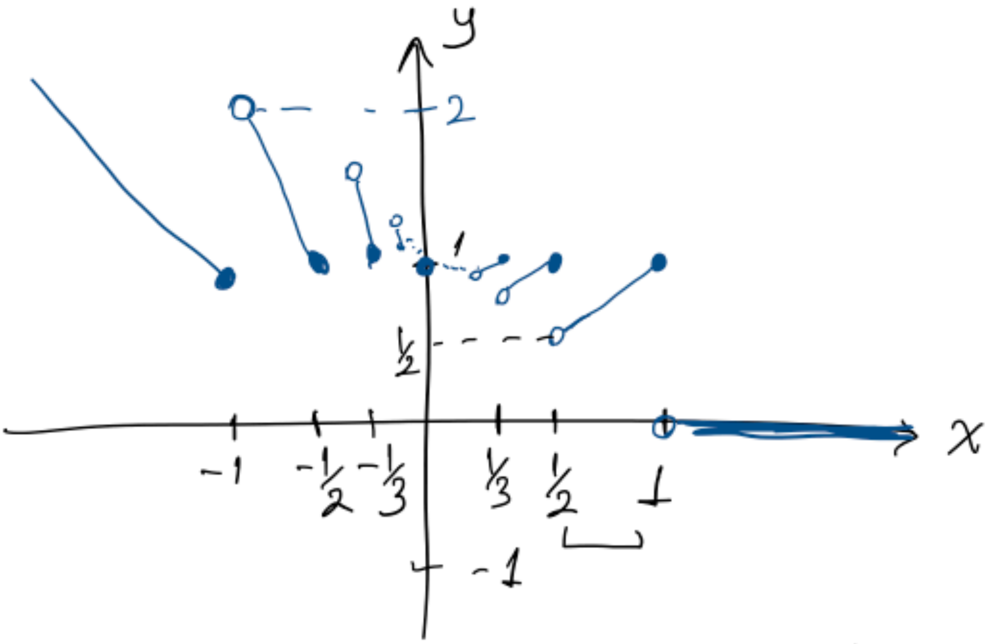
If $x \in [-\frac{1}{n}, -\frac{1}{n+1}]$ then $-\frac{1}{n} < x \leq -\frac{1}{n+1}$

$\Rightarrow f(x) = -(n+1)x$

$\Rightarrow f(x) = \ldots$

$\Rightarrow \quad \frac{1}{x} \geq -(n+1)$





If $a \in \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \right\}$ then $f(x)$ is discontinuous since limit does not exists.

$$\frac{1}{x} - 1 \leq \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x} + 1$$

Let $x > 0$, $\cancel{x}\frac{1}{\cancel{x}} - x \leq x \left\lfloor \frac{1}{x} \right\rfloor \leq \cancel{x}\frac{1}{\cancel{x}} + x$

$\lim\limits_{x \to 0^+} 1 - x = \lim\limits_{x \to 0^-} 1 + x = 1 \Rightarrow$ By squeeze thm, $\lim\limits_{x \to 0^+} f(x) = 1$

Let $x < 0$ $\cancel{x}\frac{1}{\cancel{x}} + x \leq x \left\lfloor \frac{1}{x} \right\rfloor \leq \cancel{x}\frac{1}{\cancel{x}} - x$

$\lim\limits_{x \to 0^-} 1 + x = \lim\limits_{x \to 0^-} 1 - x = 1 \Rightarrow$ By squeeze thm, $\lim\limits_{x \to 0^-} f(x) = 1.$

$$\Rightarrow \lim_{x\to 0} f(x) = 1 = f(0) \Rightarrow f \text{ is continuous at } 0.$$

Q3| Suppose that $f$ satisfies $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$ and $f(x)$ is continuous at $0$. Prove that $f$ is continuous at $a$ $\forall a \in \mathbb{R}$.

We need to show that $\lim_{x\to a} f(x) = f(a)$.

We know that $\lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to a} f(x) = \lim_{x\to 0} \underline{\underline{f(x+a)}} = \lim_{x\to 0} (f(x) + f(a)) \; *$$

Since $\lim_{x\to 0} f(x)$ & $\lim_{x\to 0} f(a)$ exist we can seperate this limit.

$$* = \lim_{x\to 0} f(x) + \lim_{x\to 0} f(a) = f(0) + f(a) = f(0+a)$$

$$= f(a)$$

since $f$ is cont at $0$

So for any $a \in \mathbb{R}$, $f(x)$ is continuous at $a$.

Q4| Let $f: [0,1] \to \mathbb{R}$ be continuous function which satisfies $f(0) = f(1)$. Prove that there exists a number

$a \in [0, \frac{1}{2}]$ so that $f(a) = f(a+\frac{1}{2})$.

Recall: Intermediate Value Theorem:

If $f(x)$ is continuous on $[a,b]$ and "s" is the number between $f(a)$ and $f(b)$, then there exists $c \in [a,b]$ so that $f(c) = s$.

Extreme Value Theorem:

If $f(x)$ is continuous on $[a,b]$, then $f(x)$ attains its maximum and minimum value on $[a,b]$.

Let $g(x) = f(x) - f(x+\frac{1}{2})$ on $[0, \frac{1}{2}]$

Observe that $g(x)$ is continuous since $f(x)$ is continuous.

We know that $f(0) = f(1)$

If $f(\frac{1}{2}) = f(1)$ then we are done since we can choose $a = \frac{1}{2}$.

Assume $f(\frac{1}{2}) \neq f(1)$. WLOG $f(\frac{1}{2}) < f(1)$

$g(0) = f(0) - f(\frac{1}{2})$ since $f(\frac{1}{2}) < f(1) = f(0)$

So $g(0) > 0$.

$f(\ ) < f(1)$

$g(\frac{1}{2}) = f(\frac{1}{2}) - f(1)$ since $f(\frac{1}{2}) <$ ...

So $g(\frac{1}{2}) < 0$

By IVT, there exists $a \in [0, \frac{1}{2}]$ so that $g(a) = 0$ which means that $f(a) - f(a + \frac{1}{2}) = 0$

$$\Rightarrow f(a) = f(a + \frac{1}{2}).$$

Q5| Given $f(x) = \begin{cases} 1 + x\cos(\frac{2\pi}{x}) & \text{if } x > 0 \\ x^2 + 1 & \text{if } x \leq 0 \end{cases}$

Show that there exists $c \in [-1, 2]$ so that $f(c) = 0$.

First, we need to show that $f(x)$ is continuous on $[-1, 2]$ to apply IVT.

$1 + x\cos(\frac{2\pi}{x})$ and $x^2 + 1$ are continuous, so we need to show continuity at $0$.

$$\lim_{x \to 0} f(x) \longrightarrow \lim_{x \to 0^-} x^2 + 1 = 1$$

$$\lim_{x \to 0} f(x) \longrightarrow \lim_{x \to 0^+} 1 + x\cos(\frac{2\pi}{x})$$

$$-1 \leq \cos(\frac{2\pi}{x}) \leq 1 \qquad x \in (0, \infty)$$

$$-x \leq x\cos(\frac{2\pi}{x}) \leq x$$

(since $x > 0$, multiplication by $x$ does not change in order of inequality

$$1 - x \le 1 + x\cos\left(\frac{2\pi}{x}\right) \le 1 + x$$

$\lim_{x\to 0^+} 1 - x = 1 = \lim_{x\to 0^+} 1 + x \Rightarrow$ By squeeze theorem,

$$\lim_{x\to 0^+} 1 + x\cos\left(\frac{2\pi}{x}\right) = 1$$

Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1$ then $\lim_{x\to 0} f(x) = 1$.

we know that $f(0) = 0 + 1 = 1$ So $f(x)$ is continuous at $0$, therefore it is continuous on $[-1, 2]$.

If $x = 2$, $f(2) = 1 + 2\underbrace{\cos(\pi)}_{-1} = -1$

If $x = -1$, $f(-1) = (-1)^2 + 1 = 2$

$0 \in [f(2), f(-1)]$ and by applying IVT, there must be a point $c \in [-1, 2]$ so that $f(c) = 0$.

**Q6** Show that the equation $f(x) = x^3 - 15x + 1$ has three roots on the interval $[-4, 4]$.

Observe that $f(x)$ is continuous since it is a polynomial.

$f(-4) = (-4)^3 - 15(-4) + 1 = -3$

$f(0) = 0 - 15 \cdot 0 + 1 = 1$

So by IVT, since $0 \in [f(-4), f(0)]$ there must be ... $f(c) = 0$.

a point $a_1 \in [-4,0]$ so that $f(a_1)=0$

We can apply IVT since $f(x)$ is cont on $[-4,0]$ which is contained in $[-4,4]$.

$f(1) = 1-19+1 = -13$

By IVT, $0 \in [f(1), f(0)]$ so $\exists a_2 \in [0,1]$ so that $f(a_2)=0$.

$f(4) = 4^3 - 19.4 + 1 = 5$

By IVT, $0 \in [f(1), f(4)]$ so $\exists a_3 \in [1,4]$ so that $f(a_3)=0$.

Observe that $a_1, a_2, a_3$ are distinct.

We conclude that there exists three real roots on $[-4,4]$.

Normally we should say $f(x)$ has at least three real roots. However in our question $f(x)$ is a polynomial of degree three so it can have maximum three real roots.

Q7| Show that if $f$ is continuous function defined on a closed interval $[a,b]$ then the range of $f(x)$ is a closed interval.

is also a closed interval

Since $f$ is continuous on $[a,b]$ then by EVT $f(x)$ must have maximum & minimum values on this interval, call them $f_{min}$, $f_{max}$.

By IVT, we know that for any $t \in [f_{min}, f_{max}]$ there must be $m \in [a,b]$ so that $f(m) = t$.

So the range is a closed interval of the form $[f_{min}, f_{max}]$.

**Q8** Determine whether the following statements are true or false

a) Let $f$ and $g$ be two functions on $\mathbb{R}$. If $f$ and $g$ are discontinuous at $b$, then $f+g$ is discontinuous at $b$

FALSE

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases} \qquad g(x) = \begin{cases} -1 & x > 0 \\ 1 & x \leq 0 \end{cases}$$

Observe that $f(x)$ and $g(x)$ are discontinuous at $0$

$(f+g)(x) = 0$ which is continuous at $0$

b) If $f(x)$ is continuous everywhere, then $|f(x)|$ is continuous everywhere.

TRUE

If $\lim_{x\to a} f(x) = L$ then $\lim_{x\to a} |f(x)| = |L|$

Since $f(x)$ is continuous everywhere $L = f(a)$

then $\lim_{x\to a} |f(x)| = |f(a)|$ which means that

$|f(x)|$ is continuous everywhere.

c) If $\lim_{x\to c} f(x)$ and $f(c)$ exist, then $f(x)$ is continuous at $c$.

FALSE $\quad f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$

$\lim_{x\to 0} f(x) = 1 \neq f(0) \Rightarrow f$ is discontinuous at $0$.

d) If a function $f : [153, \infty) \longrightarrow \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = 2021$, then $f$ is bounded.

TRUE

$\lim_{x\to\infty} f(x) = 2021 \Rightarrow \forall \varepsilon > 0 \ \exists M$ s.t $\forall x > M \ |f(x) - 2021| < \varepsilon$

Let $\varepsilon = 1$, then $\exists M_1 > 153 \ \forall x > M_1, \ |f(x) - 2021| < 1$
$2020 < f(x) < 2022$

... there

$f(x)$ is continuous on $[153, M_1]$ so by EVT ...
exist maximum and minimum values call them $t, s$.
max & min.

$\Rightarrow$ $f$ is bounded above by $\max\{2022, t\}$
below by $\min\{2020, s\}$

$\Rightarrow$ $f$ is bounded.