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Q1 If $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2} = -2$, find $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$

Recall: If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$

then $\lim\limits_{x\to a} f(x).g(x) = L.M$

We know that $\lim\limits_{x\to 0} x^2 = 0$

So by using the property above we can say

$$\lim_{x\to 0} x^2 \cdot \frac{f(x)}{x^2} = \underbrace{\lim_{x\to 0} x^2}_{0} \cdot \underbrace{\lim_{x\to 0} \frac{f(x)}{x^2}}_{-2} = 0$$

$$\| $$

$$\lim_{x\to 0} f(x)$$

Again by the same property,

$$\lim_{x\to 0} x \cdot \frac{f(x)}{x^2} = \underbrace{\lim_{x\to 0} x}_{0} \cdot \underbrace{\lim_{x\to 0} \frac{f(x)}{x^2}}_{-2} = 0$$

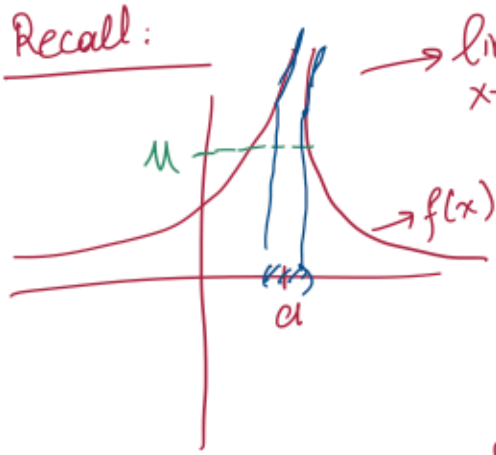
$$\| $$

$$\lim_{x\to 0} \frac{f(x)}{x}$$

Q2| Prove the following limits

a) $\lim\limits_{x \to 1^+} \dfrac{1}{1-x} = -\infty$

Recall:

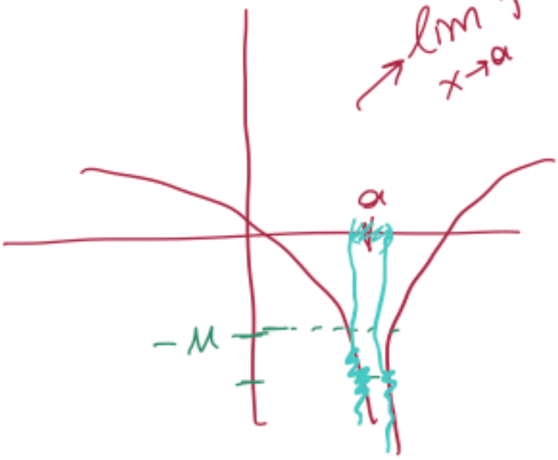


$\lim\limits_{x \to a} f(x) = \infty$

$\forall M > 0 \; \exists \delta > 0$ so that

$0 < |x-a| < \delta$ implies

$f(x) > M$



$\lim\limits_{x \to a} f(x) = -\infty$

$\forall M > 0 \; \exists \delta > 0$ so that

$0 < |x-a| < \delta$ implies

$f(x) < -M$

$\underline{\underline{\lim\limits_{x \to 1^+}}} \dfrac{1}{1-x} = -\infty \iff \forall M > 0 \; \exists \delta > 0$ so that

$0 < x-1 < \delta$ implies that

$f(x) < -M$

Let $M > 0$ be given. Choose $\delta = \cdots \frac{1}{M}$

We know that $0 < x-1 < \delta$

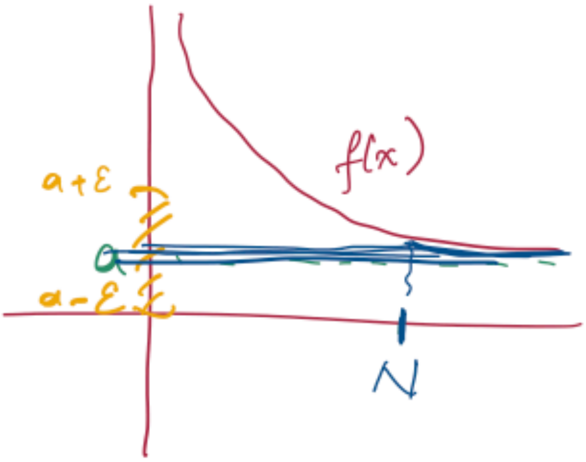
We want to find M values which gives

$f(x) < -M$

$f(x) = \frac{1}{1-x}$ From $0 < x-1 < \delta$, we have

$$\frac{1}{x-1} > \frac{1}{\delta} \Rightarrow \frac{1}{1-x} < \underbrace{-\frac{1}{\delta}}_{-M} = -M$$

b) $\lim\limits_{x \to \infty} \frac{1}{(1+x)^2} = 0$



$\lim\limits_{x \to \infty} f(x) = a$

$\Longleftrightarrow$

$\forall \varepsilon > 0 \quad \exists N > 0$ so that

for every $x > N$ implies

$|f(x) - a| < \varepsilon$

$$\lim_{x \to \infty} \frac{1}{(1+x)^2} = 0 \Longleftrightarrow \forall \varepsilon > 0 \ \exists N > 0 \text{ so that } \forall x > N \Rightarrow \left| \frac{1}{(1+x)^2} - 0 \right| < \varepsilon$$

Let ε be given. Choose $N = \frac{1}{\sqrt{\varepsilon}}$

We know that $x > N \Rightarrow x+1 > N+1 > 1$

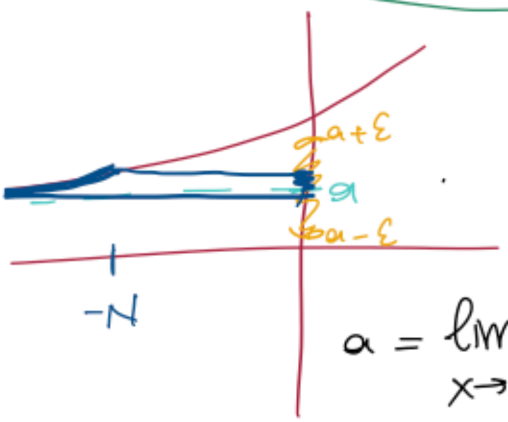
$$\Rightarrow (x+1)^2 > (N+1)^2$$

$$\Rightarrow \frac{1}{(N+1)^2} > \frac{1}{(x+1)^2}$$

$$\Rightarrow \left|\frac{1}{(x+1)^2}\right| < \underbrace{\frac{1}{(N+1)^2}}_{\varepsilon} < \underbrace{\frac{1}{N^2}}_{\varepsilon} = \varepsilon$$

$\rightarrow$ $N^2+2N+1 > N^2 \Rightarrow \dfrac{1}{N^2} > \dfrac{1}{(N+1)^2}$

since $N>0$

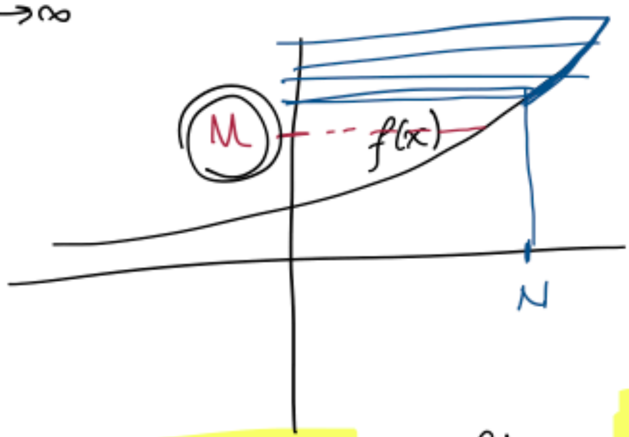


$\Leftrightarrow$ $\forall \varepsilon>0 \quad \exists N>0$ so that $\forall x<-N$ we will have $|f(x)-a|<\varepsilon$

$$a = \lim_{x\to-\infty} f(x)$$

c) $\displaystyle\lim_{x\to\infty} \sqrt{x} = \infty$

$\displaystyle\lim_{x\to\infty} f(x) = \infty \Leftrightarrow \forall M>0 \quad \exists N>0$ so that $\forall x>N$ we have $f(x)>M$



Let M be given. Choose $N = \ldots M^2$

We know that $\boxed{x>N}$ and we want to find a bound for $\sqrt{x}$

$x > N \Rightarrow \sqrt{x} > \sqrt{N} = M$

$\sqrt{N} = M$

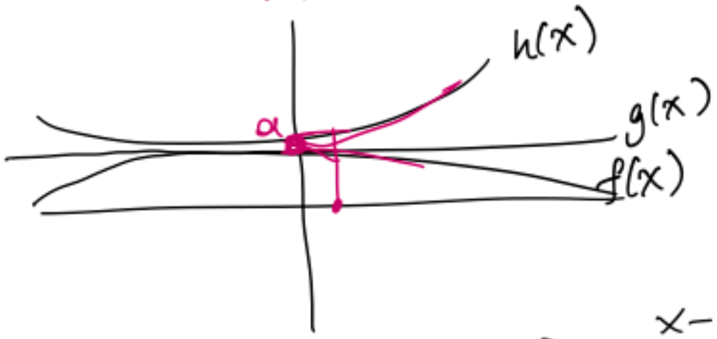
Q3| Find the following limits, if exists

a) $\lim\limits_{x \to \infty} \dfrac{x + \sin^3 x}{5x+6}$

Recall: Squeeze Theorem: Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some interval containing except possibly at a.

Also suppose that $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to a} h(x) = L$

Then $\lim\limits_{x \to a} g(x) = L$



$-1 \leq \sin^3 x \leq 1 \Rightarrow \underbrace{\dfrac{x-1}{5x+6}}_{f(x)} \leq \dfrac{x+\sin^3 x}{5x+6} \leq \underbrace{\dfrac{x+1}{5x+6}}_{h(x)}$

$\dfrac{x-1}{5x+6} \leq \dfrac{x+\sin^3 x}{5x+6} \leq \dfrac{1+x}{5x+6}$

(we can divide by $5x+6$ since we can choose x values which makes $5x+6 > 0$

$$\lim_{x\to\infty} \frac{x-1}{9x+6} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{9+\frac{6}{x}} = \frac{1}{9} = \lim_{x\to\infty} f(x)$$

$$\lim_{x\to\infty} \frac{x+1}{9x+6} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{9+\frac{6}{x}} = \frac{1}{9} = \lim_{x\to\infty} h(x).$$

By squeeze theorem, since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \frac{1}{9}$

we have $\lim_{x\to\infty} g(x) = \frac{1}{9}$.

b) $\lim_{x\to 0} \frac{\tan^2 x + \sin x^3}{2x^2}$

$$\lim_{x\to 0} \frac{\tan^2 x}{2x^2} = \lim_{x\to 0} \frac{1}{2} \cdot \frac{\sin^2 x}{x^2} \cdot \frac{1}{\cos^2 x}$$

We know that $\underline{\underline{\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = 1}}$ & $\underline{\underline{\lim_{x\to 0} \frac{1}{2} \cdot \frac{1}{\cos^2 x} = \frac{1}{2}}}$

Since both limits exist, we have

$$\underline{\lim_{x\to 0} \frac{\tan^2 x}{2x^2}} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\sin x^3}{2x^2} = \lim_{x\to 0} \frac{1}{2} \cdot x \, \frac{\sin x^3}{x^3}$$

We know that $\lim_{x\to 0} \frac{1}{2} x = 0$ & $\lim_{x\to 0} \frac{\sin x^3}{x^3} = 1$

Since both limits exist, we have $x \to 0 \Rightarrow x^3 \to 0$

$$\lim_{x\to 0} \frac{\sin x^3}{2x^2} = 0.1 = \underline{\underline{0}}$$

Overall, limits of $\frac{\tan^2 x}{2x^2}$ and $\frac{\sin x^3}{2x^2}$ exist we can use the property to find limit of

$$\frac{\tan^2 x + \sin x^3}{2x^2}$$

Hence we have $\lim_{x\to 0} \frac{\tan^2 x + \sin x^3}{2x^2} = 0 + \frac{1}{2} = \frac{1}{2}$.

! $\lim_{x\to 0} \frac{\sin x}{x} = 1 \iff \lim_{f(x)\to 0} \frac{\sin(f(x))}{f(x)} = 1$

while $x \to 0$

c) $\lim_{x\to -\infty} 2x - \sqrt{4x^2-3x}$

$$\lim_{x\to -\infty} \left(2x - \sqrt{4x^2-3x}\right) \frac{\left(2x + \sqrt{4x^2-3x}\right)}{2x + \sqrt{4x^2-3x}}$$

$$\boxed{\lim_{x\to -\infty} \frac{4x^2 - |4x^2-3x|}{2x + \sqrt{4x^2-3x}}} = \lim_{x\to -\infty} \frac{\cancel{4x^2} - \cancel{4x^2} + 3x}{2x + \sqrt{4x^2-3x}}$$

$>0 \; (4x^2) \geq (3x) < 0 \quad \forall x \in \mathbb{R} - (0,1)$

$$\lim_{x\to -\infty} \frac{3}{2 + \sqrt{4 - \frac{3}{x}}} = \frac{3}{2 + \sqrt{4}} = \frac{3}{4}$$

! Do not forget absolute value.

d) $\lim\limits_{x\to\infty} (x+\cos x)$

We know that $\quad -1 \le \cos x \le 1$

$$x-1 \le x+\cos x \le x+1.$$

$$\lim_{x\to\infty} x-1 = \infty \Rightarrow \lim_{x\to\infty} x+\cos x = \infty$$

since $x-1$ is a lower bound for $x+\cos x$

e) $\lim\limits_{x\to2} \dfrac{\sqrt{4-4x+x^2}}{x-2}$

$$\lim_{x\to2} \frac{\sqrt{(x-2)^{\cancel{2}}}}{x-2} = \lim_{x\to2} \frac{|x-2|}{x-2}$$

$$\lim_{x\to2^+} \frac{|x-2|}{x-2} = \lim_{x\to2^+} \frac{\cancel{x-2}}{\cancel{x-2}} = 1$$

$$\ne \Rightarrow \lim_{x\to2} \frac{|x-2|}{x-2} \ \underline{\underline{\text{dne}}}$$

$$\lim_{x\to2^-} \frac{|x-2|}{x-2} = \lim_{x\to2^-} \frac{2-x}{x-2} = -1$$

f) $\lim\limits_{x\to8} \dfrac{x^{2/3}-4}{x^{1/3}-2} = \lim\limits_{x\to8} \dfrac{\cancel{(x^{1/3}-2)}(x^{1/3}+2)}{\cancel{x^{1/3}-2}}$

$$= \lim_{x\to8} (x^{1/3}+2) = \underline{\underline{4}}$$

$$\lim_{y\to2} \frac{y^2-4}{y-2}$$

$y=x^3 \quad x\to8 \Rightarrow y\to2$

g) $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+25} - 5}{x^2}$

$$\lim_{x \to 0} \frac{(\sqrt{x^2+25} - 5)}{x^2} \cdot \frac{(\sqrt{x^2+25} + 5)}{(\sqrt{x^2+25} + 5)}$$

$$\shortparallel$$

$$\lim_{x \to 0} \frac{|x^2+25| - 25}{x^2 \cdot (\sqrt{x^2+25} + 5)} = \lim_{x \to 0} \frac{\cancel{x^2} + \cancel{25} - \cancel{25}}{\cancel{x^2}(\sqrt{x^2+25} + 5)}$$

since $x^2 > 0$
$x^2+25 > 0$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x^2+25} + 5} = \frac{1}{\sqrt{25} + 5} = \frac{1}{10}$$

Q4| Let the function f defined as follows

$$f(x) = \begin{cases} ax^2 + 1 & \text{if } x < 0 \\ \underline{\underline{x + b}} & \text{if } x \geq 0 \end{cases}$$

Find a and b values which makes $f(x)$ continuous everywhere.

$\lim\limits_{x \to a} f(x) = f(a)$ then f is called continuous at a.

We know that ax^2+1, $x+b$ are continuous functions for all $a, b \in \mathbb{R}$. Only point we need to check is 0.

$\ldots = f(0)$

$\lim_{x\to 0} f(x) =$ ...

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+b = b$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ax^2+1 = 1$$

To make this limit exist $b=1$.

$$f(x) = \begin{cases} ax^2+1 & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$$

$\underbrace{\lim_{x\to 0} f(x)}_{1} = \underbrace{f(0)}_{1}$ Therefore any value for a would work.

$\therefore$ If $b=1$, and a is an arbitrary real number then f will be continuous everywhere.

Q5| a) Prove that if $\lim_{x\to 0} \frac{f(x)}{x} = L$, then $\lim_{x\to 0} \frac{f(bx)}{x} = b.L$ for $b \neq 0$

Observe that $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{bx\to 0} \frac{f(bx)}{bx} = L$

since $x\to 0 \Rightarrow bx \to 0$

... this fact we can write $\lim b.\frac{f(bx)}{bx} = b.L$

By using this ... $x \to 0$ ...

since $\lim_{x \to 0} b$ exists.

b) What if $b=0$?

Let $b=0$ then $\lim_{x \to 0} \frac{f(0 \cdot x)}{x} = \lim_{x \to 0} \frac{f(0)}{x}$

Existence of limit depends on the value $f(0)$

If $f(0)=0$ then we have $\lim_{x \to 0} \frac{0}{x} = 0$

If $f(0) \neq 0$ then $\lim_{x \to 0} \frac{f(0)}{x}$ dne

c) By using part a, evaluate $\lim_{x \to 0} \frac{\sin(2x)}{x}$

$f(x)$ will be $\sin x$.

So $f(2x) = \sin(2x)$

By using $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and part a we have $\lim_{x \to 0} \frac{\sin(2x)}{x} = 2 \cdot 1 = 2$.