Rec-4

12 Kasım 2021 Cuma 13:35

Quiz 1: Given that $f(x)=|x-153|-|x+153|+6x$

a) Solve the inequality $f(x)<0$

b) Determine whether f is odd, even or neither and explain.

c) By using the information you get from part a & b ONLY conclude that the solution set of the equation $f(x)=0$ is $\{0\}$.

a) $f(x)=|x-153|-|x+153|+6x<0$

Case 1: $x\in(-\infty,-153)$

$$f(x) = -\cancel{x}+153+\cancel{x}+153+6x<0$$

$$6x<-306 \Rightarrow x\in(-\infty,-51)$$

Soln interval for case $1=(-\infty,-51)\cap(-\infty,-153)$
$$= \boxed{(-\infty,-153)}$$

Case 2: $x\in[-153,153]$

$$f(x) = -x+\cancel{153}-x-\cancel{153}+6x<0 \Rightarrow 4x<0$$

Sol^n interval for case $2=[-153,153]\cap(-\infty,0)$

$\overset{\shortparallel}{[-193,0)}$

Case 3: Similarly, you can find the solution interval for $x > 193$ is $\emptyset$.

Overall, Solution set is $(-\infty, 0)$ for $f(x) < 0$

b) $f(-x) = |-x-193| - |-x+193| - 6x$

$= |x+193| - |x-193| - 6x$

$|a| = |-a|$ $\quad = -(|x-193| - |x+193| + 6x) = -f(x)$

$\Rightarrow$ Since $f(-x) = -f(x)$ we can say $f(x)$ is $\underline{\underline{\text{odd}}}$

c) we know that $f(x)$ is odd, so graph of $f(x)$ is symmetric with respect to origin.

From part a, $f(x) < 0 \iff x < 0$

since graph is symmetric wrt origin we can say $f(x) > 0 \iff x > 0$

Observe that $f(0) = 0$, so the only solution for $f(x) = 0$ is $\{0\}$.

Q1 | Use the formal definition to prove the followings

a) $\lim_{x \to -2} 3x+1 = -5$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ 0 < |x-a| < \delta$ implies that $|f(x) - L| < \varepsilon$

For this question, we should find $\delta > 0$ for a given arbitrary $\varepsilon > 0$ so that $\underline{0 < |x-a| < \delta} \Rightarrow |f(x) - L| < \varepsilon$

Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{3}$

We know that $0 < |x-(-2)| < \delta$

$|x+2| < \delta$ (*)

(We need to show this (*) will imply $|3x+1-(-5)| < \varepsilon$)

$|3x+1-(-5)| = |3x+6| = 3|x+2| < 3\delta = \varepsilon$

$\delta = \frac{\varepsilon}{3}$

b) $\lim_{x \to -1} x^2 - 5x + 8 = 14 \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ 0 < |x+1| < \delta$ implies $|x^2 - 5x + 8 - 14| < \varepsilon$

Let $\varepsilon > 0$ be given. Choose $\delta = \ldots$

We know that $\underline{0 < |x+1| < \delta}$

(Next we should show $|x^2 - 5x - 6| < \varepsilon$)

$$|x^2-5x-6| = |(x-6)(x+1)| = |x-6|\underbrace{|x+1|}_{<\delta} < |x-6|\delta$$

Now to get rid of $|x-6|$ we can use $|x+1|<\delta$

$$|x-6| = |x+1-7| \le |x+1| + |-7| = \underbrace{|x+1|}_{<\delta} + 7 < \delta + 7$$

$|a+b| \le |a| + |b|$

$$\Rightarrow |x-6| < \delta + 7 \quad (*)$$

! Note that we can use (*) and we will get $|x^2-5x-6| < \delta^2 + 7\delta$. Since $\varepsilon = \delta^2 + 7\delta$ is not an easy thing to express as $f(\varepsilon) = \delta$ we should do some further simplifications on δ bound.

Now assume that $\delta \le 1$ then $\delta + 7 \le 8$

We already conclude that $|x^2-5x-6| < |x-6|\delta$

$$< \underbrace{(\delta+7)}\delta$$

$$< 8\delta = \varepsilon$$

$$\delta = \frac{\varepsilon}{8}$$

Since $\delta \le 1 \Rightarrow \frac{\varepsilon}{8} \le 1 \Rightarrow \varepsilon \le 8$

So For $\varepsilon \le 8$ we can choose $\delta = \frac{\varepsilon}{8}$

For $\varepsilon > 8$ choose $\delta = 1$

$\forall \varepsilon > 0$ choose $\delta = \min\left\{1, \frac{\varepsilon}{8}\right\}$ so that

$0 < |x+1| < \delta$ implies $|x^2 - 5x + 8 - 14|$

$$= |x-6|\,|x+1|$$

if $\varepsilon > 8$, $\delta = 1$

$$|x-6|\,|x+1| < (\delta+7)\cdot\delta < 8 < \varepsilon$$

if $\varepsilon \le 8$, $\delta = \frac{\varepsilon}{8}$

$$|x-6|\,|x+1| < (\delta+7)\delta < 8\delta = \varepsilon$$

since $\delta \le 1$

we know that $\delta = \frac{\varepsilon}{8}$

& $\varepsilon \le 8 \Rightarrow \delta \le 1$

$\delta + 7 \le 8$

c) $\lim\limits_{x\to 1} (3x+1) \neq 5$

$$\lim_{x\to a} f(x) = L \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ 0 < |x-a| < \delta$$
implies $|f(x) - L| < \varepsilon$

$$\neg\Big(\forall \varepsilon > 0 \ \exists \delta > 0 \ \underline{\underline{\forall x}} \big(\underbrace{0 < |x-a| < \delta}_{p} \Rightarrow \underbrace{|f(x) - L| < \varepsilon}_{q}\big)\Big)$$

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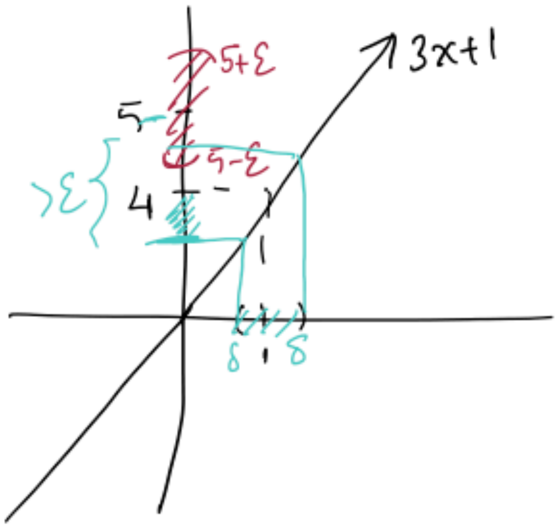
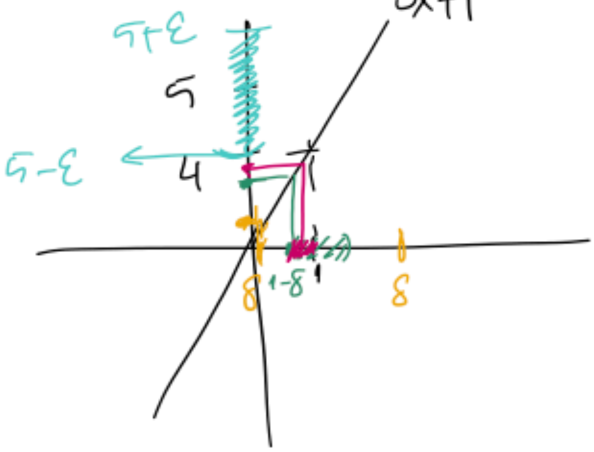
$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \ \big(\underbrace{0 < |x-a| < \delta}_{p} \wedge \underbrace{|f(x) - L| \ge \varepsilon}_{\neg q}\big)$$

$$\neg(p \Rightarrow q) \equiv p \wedge \neg q$$



Choose $\varepsilon = 1$



Choose $\varepsilon = 1$ and δ will be arbitrary positive number.

$$0 < |x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta$$
$$\Rightarrow x \in (1-\delta, 1+\delta)$$

Now choose $x_0 = 1 - \frac{\delta}{2} \in (1-\delta, 1+\delta)$

and observe that $|f(x_0) - 5| = \left|3 - \frac{3\delta}{2} + 1 - 5\right|$

$$= \left|-1 - \frac{3\delta}{2}\right|$$

$\frac{3\delta}{2} > 0$

$|-a| = |a|$ $\quad = \left|1 + \frac{3\delta}{2}\right| > 1 = \varepsilon$

Overall we showed that when $\varepsilon = 1$, there exists a point $x_0 = 1 - \frac{\delta}{2}$ for any $\delta > 0$ we have

$$|f(x_0) - 5| > \varepsilon \quad \text{where} \quad \underbrace{x_0 \in (1-\delta, 1+\delta)}_{(|x_0 - 1| < \delta)}$$

(Try to choose $\varepsilon = \frac{1}{2}$) $\leftarrow$

d) $\lim_{x \to 0} \sin \frac{1}{x} \neq a$ for any $a \in \mathbb{R}$.

Case 1: Assume that $a \geq 0$.

Need to show: find a point $\underbrace{x_0 \in (-\delta, \delta)}_{(0 < |x_0| < \delta)}$ so that

$\exists \varepsilon \quad \forall \delta \quad |f(x_0) - a| \geq \varepsilon$

Choose $\varepsilon = 1$. We want to make $\left|\sin \frac{1}{x_0} - a\right| \geq 1$ for $x_0 \in (-\delta, \delta)$

$-1 \leq \sin \frac{1}{x_0} \leq 1$ We know that $\exists n \in \mathbb{N}$ so that

$$x_0 = \frac{1}{\frac{3\pi}{2} + 2n\pi} < \delta \Rightarrow x_0 \in (-\delta, \delta)$$

$$\sin \frac{1}{x_0} = -1 \Rightarrow \left|\sin \frac{1}{x_0} - a\right| = |-1-a| = |1+a| > 1 = \varepsilon$$

since $a \geq 0$

Case 2: Assume $a < 0$.

We want to find x_0 so that $\left|\sin \frac{1}{x_0} - a\right| \geq 1$

Let choose $\varepsilon = 1$ again.

If we can fix $\sin \frac{1}{x_0} = 1 \quad \forall \delta$

$$\frac{1}{x_0} = \frac{\pi}{2} + 2n\pi, \text{ for some } n \in \mathbb{N}$$

$$\rightarrow x_0 = \frac{1}{\frac{\pi}{2} + 2n\pi}$$

We need to be careful since $x_0 \in (-\delta, \delta)$

We can find $N \in \mathbb{N}$ so that $\frac{1}{\frac{\pi}{2} + 2N\pi} < \delta$

so we have $x_0 = \frac{1}{\frac{\pi}{2} + 2N\pi} \in (-\delta, \delta)$

so that $\left|\sin \frac{1}{x_0} - a\right| = |1-a| \geq 1 = \varepsilon$

$a<0$

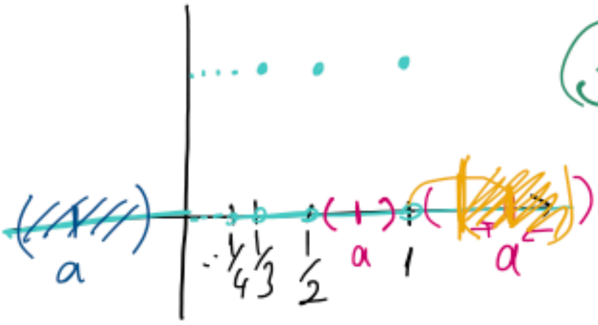
Q2 Let $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

a) Show that if $a \neq 0$ then $\lim_{x \to a} f(x) = 0$.

Assume that $a \neq 0$.

Let $\varepsilon > 0$ be given. Choose $\delta = \dots$

we know that $0 < |x-a| < \delta$



(1) If $a > 1$ then choose $\delta = \frac{a-1}{2}$

So any $x \in (a-\delta, a+\delta)$ we have $f(x) = 0$

So $|f(x) - 0| = |0-0| < \varepsilon$

(2) If $a < 0$ then choose $\delta = \frac{-a}{2}$

So $f(x) = 0 \quad \forall x \in \left(\frac{3a}{2}, \frac{a}{2}\right)$

Therefore $|f(x) - 0| = |0-0| < \varepsilon$

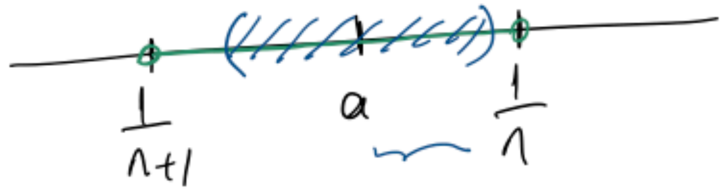
(3) If $0 < a < 1$ then $\frac{1}{n+1} < a < \frac{1}{n}$ for some $n \in \mathbb{N}$.

$\forall \varepsilon > 0$ choose $\delta = \min\left\{ a - \frac{1}{n+1}, \frac{1}{n} - a \right\}$

then $f(x) = 0 \quad \forall x \in (a-\delta, a+\delta)$

... makes $|f(x) - 0|$

which makes ...



$$|0| < \varepsilon$$

So we can conclude that $\lim\limits_{x \to a} f(x) = 0$

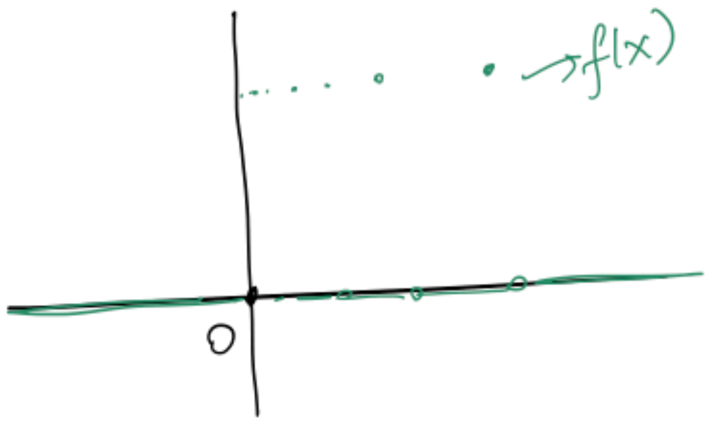
when $a \neq 0$.

(4) Exercise show that $\lim\limits_{x \to 1} f(x) = 0$

b) Show that $\lim\limits_{x \to 0} f(x)$ dne.

we should show $\lim\limits_{x \to 0} f(x) \neq L$

where $L \in \mathbb{R}$.



__Case 1:__ Assume that $L \neq 1$.

Choose $\varepsilon = |L - 1|$ and δ arbitrary.

we know that $\exists n \in \mathbb{N}$ so that $\frac{1}{n} < \delta$

Choose $x_0 = \frac{1}{n} \Rightarrow |f(x_0) - L| = \underbrace{|1 - L|}_{\varepsilon}$

$$\Rightarrow |f(x_0) - L| \geq \varepsilon.$$

__Case 2:__ Assume that $L = 1$.

Choose $\varepsilon = 1$ and δ will be arbitrary. ... $\forall n \in \mathbb{N}$

Choose $\varepsilon = 1$ ... so that $x_0 = \frac{1}{n}$.

we know that $\exists x_0 \in (-\delta, \delta)$

So $|f(x_0) - L| = |0 - 1| = |1| \geq 1 = \varepsilon$

$\Rightarrow$ Hence $\lim_{x \to 0} f(x)$ does <u>not</u> exist.

Q3| Let $\lim_{x \to 1} f(x) = 10$ and $\lim_{x \to 1} g(x) = 20$. Find.

a) $\lim_{x \to 1} (f(x))^2$

Since $\lim_{x \to 1} f(x)$ <u>exists</u> then $\underbrace{\underset{=10}{\lim_{x \to 1} f(x)} \cdot \underset{=10}{\lim_{x \to 1} f(x)}}_{100}$

$= \lim_{x \to 1} (f(x))^2 = 100$

b) $\lim_{x \to 1} (g(x) - 10) =$ exercise

c) $\lim_{x \to 1} 5(g(x) - 10) =$ exercise

Q4| Determine whether the following statements are true or false

a) If $\lim_{x \to 0} f(x)^2 = 9$ then $\left(\lim_{x \to 0} f(x)\right)^2 = 9$.

<u>FALSE</u>

Let $f(x) = \begin{cases} 3 & \text{if } x > 0 \\ -3 & \text{if } x < 0 \end{cases}$ then $f(x)^2 = 9$

... line

$\lim_{x \to 0} f(x)^2 = 9$ but $\lim_{x \to 0} f(x)$ dne.

b) If $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ does not exist, then $\lim_{x \to a} (f+g)(x)$ does not exist.

<u>TRUE</u>

Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} (f+g)(x)$ exists and $\lim_{x \to a} g(x)$ dne.

By limit properties, $\lim_{x \to a} (f+g)(x) - \lim_{x \to a} f(x)$

since both limits exist $\leftarrow$ $=$

$\lim_{x \to a} (f+g-f)(x)$

$=$

$\lim_{x \to a} g(x)$

$\Rightarrow \lim_{x \to a} g(x)$ exist $\lightning$ $\leftarrow$ This is a contradiction

So our assumption is not true so $\lim_{x \to a} (f+g)(x)$ does not exist.

c) If $\lim_{x \to 0} f(x)$ exists and $\lim_{x \to 0} g(x)$ does not exist, then $\lim_{x \to 0} f(x) \cdot g(x)$ does not exist.

<u>FALSE</u>: Let $f(x) = x$ & $g(x) = \frac{1}{x}$

then $\lim f(x) = 0$ & $\lim_{x \to 0} g(x)$ dne

$x \to 0$

Observe that $f(x) \cdot g(x) = x \cdot \frac{1}{x} = 1$

$\lim_{x \to 0} 1 = 1$ so limit of $f(x) \cdot g(x)$ exists.

d) If $\lim_{x \to 0} f(x) = L$ for some $L \in \mathbb{R}$, then $\lim_{x \to 0} f(x^2) = L$

TRUE

$\lim_{x \to 0} f(x) = L \iff \forall \varepsilon > 0 \; \exists \delta_0 > 0$ s.t $0 < |x| < \delta_0 \Rightarrow |f(x) - L| < \varepsilon$.

Choose $\delta_0 \leq 1$ for $\lim_{x \to 0} f(x) = L$

(we can make this choice since if $\delta > 1$ then any number less than δ also works)

So we have $0 < |x| < \delta_0 \implies 0 < |x^2| < \delta_0^2 \leq \delta_0$

since $\delta_0 \in (0, 1)$

which means that $x^2 \in (-\delta_0, \delta_0)$

So $|f(x^2) - L| < \varepsilon$

$\implies \lim_{x \to 0} f(x^2) = L$.

e) If $\lim_{x \to 0} f(x^2) = L$ then $\lim_{x \to 0} f(x) = L$.

FALSE Let $f(x) = \begin{cases} 3 & \text{if } x \geq 0 \\ -3 & \text{if } x < 0 \end{cases}$

then $f(x^2) = 3 \quad \forall x$

So $\lim_{x \to 0} f(x^2) = 3$ but $\lim_{x \to 0} f(x)$ d ne.

Q9| What is the domain of $x^3\cos\left(\frac{193}{x}\right)$? Use

formal definition to prove that $\lim_{x\to 0} x^3\cos\left(\frac{193}{x}\right) = 0$.

Domain of $x^3\cos\left(\frac{193}{x}\right) = \mathbb{R}-\{0\}$

Let ε be given. Choose $\delta = \sqrt[3]{\ldots}$

$0<|x|<\delta$ implies $\left|x^3\cos\left(\frac{193}{x}\right)-0\right| \le |x^3| \le \underset{\varepsilon}{\underset{\shortparallel}{\delta^3}}$

$$\left|\cos\left(\frac{193}{x}\right)\right| \le 1$$

5. Suppose that $\lim_{x\to 5} \frac{f(x)}{x} = 3$ and $\lim_{x\to 5} g(x) = 4$.

Find the following limits.

(a) $\lim_{x\to 5} f(x)$.

Solution:

since each limit exists.

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{f(x)}{x} \cdot x = \lim_{x\to 5} \frac{f(x)}{x} \cdot \lim_{x\to 5} x$$

$$= 3 \cdot 5$$

$$= 15.$$

(b) $\lim_{x\to 5} \frac{g(x)}{f(x)-1}$

From part (a), we have $\lim_{x\to 5} f(x) = 15$.

Therefore $\lim_{x\to 5} (f(x) - 1) = 14$. Then

$$\lim_{x\to 5} \frac{g(x)}{f(x)-1} = \frac{\lim_{x\to 5} g(x)}{\lim_{x\to 5} (f(x)-1)} = \frac{4}{14} \checkmark$$

Since each limit exists.