Rec-2

27 Ekim 2021 Çarşamba 13:33

Q1| Find the domain and range of each function and sketch their graphs.

(a) $f(x) = \lfloor x^2 \rfloor$ where $\lfloor . \rfloor$ is the greatest integer function.

Recall: $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$

Domain of $f(x) = \mathbb{R}$

$f(x) = \lfloor x^2 \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x^2\}$

since $x^2 \geq 0$, $f(x) \geq 0 \Rightarrow$ range of $f(x) = \mathbb{Z}^{\geq 0}$

nonnegative integers

Observe that $m \leq x^2 < m+1$

by taking square root of each side; $\sqrt{m} \leq |x| < \sqrt{m+1}$

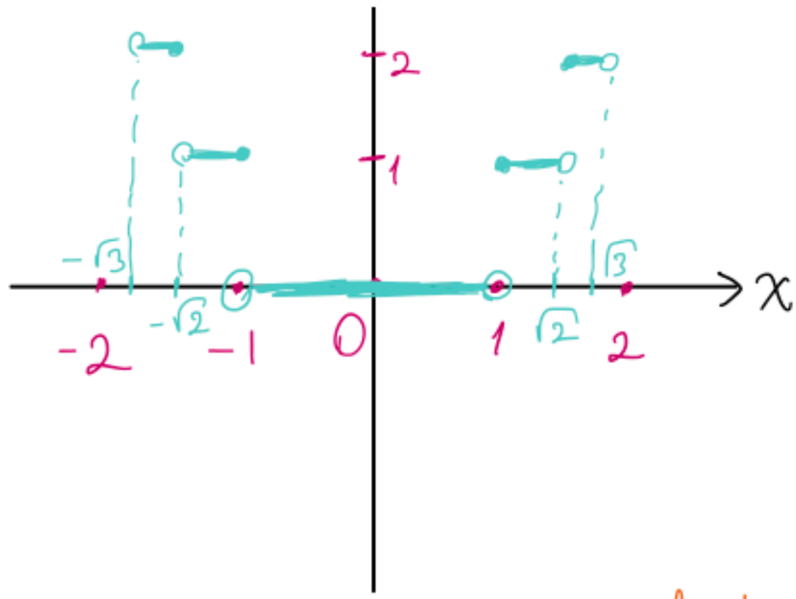
For $m=0 \Rightarrow 0 \leq |x| < \sqrt{1} = 1$

$\Rightarrow f(x) = 0$ for $x \in (-1, 1)$

For $m=1 \Rightarrow 1 \leq x < \sqrt{2}$

$\Rightarrow f(x) = 1$ for $x \in [1, \sqrt{2}) \cup (-\sqrt{2}, -1]$

In general, $f(x) = m$ for $x \in [\sqrt{m}, \sqrt{m+1}) \cup (-\sqrt{m+1}, -\sqrt{m}]$



b) $f(x) = \lceil \sin x \rceil$ where $\lceil \cdot \rceil$ is the least integer function

Recall: $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$

Domain of $f(x) = \mathbb{R}$

$f(x) = \lceil \sin x \rceil = \min\{n \in \mathbb{Z} \mid n \geq \sin x\}$

We know that $-1 \leq \sin x \leq 1$ and also

$n-1 < x \leq n$ for $\lceil x \rceil = n$

If we combine these information, range of $f(x) = \{-1, 0, 1\}$

$f^{-1}(-1) = \{x \in \mathbb{R} \mid -2 < \sin x \leq -1\}$

since $-1 \leq \sin x \leq 1$

$= \{x \in \mathbb{R} \mid \sin x = -1\}$

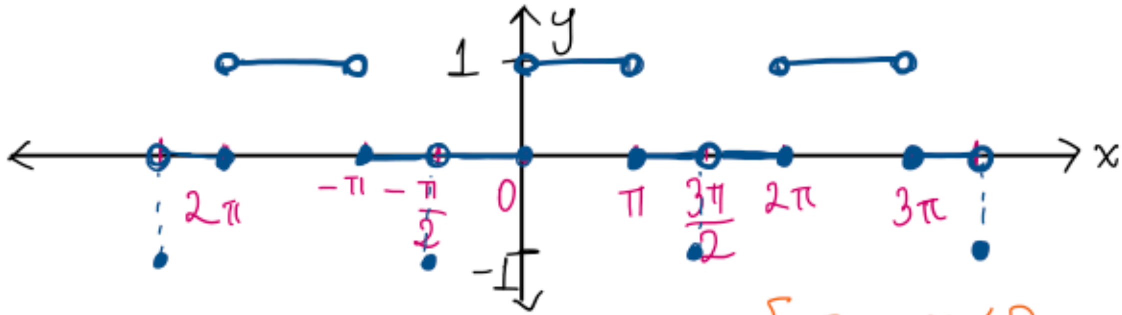
$= \{x \in \mathbb{R} \mid x = -\frac{\pi}{2} + 2\pi k \text{, for some } k \in \mathbb{Z}\}$

$f^{-1}(0) = \{x \in \mathbb{R} \mid -1 < \sin x \leq 0\}$

$$= \left\{ x \in \mathbb{R} \mid x \in \left(-\frac{\pi}{2} + 2\pi k, 2\pi k\right] \cup \left[(k+1)\pi, -\frac{\pi}{2} + 2\pi(k+1)\right) \right\}$$

$$f^{-1}(1) = \{ x \in \mathbb{R} \mid 0 < \sin x \leq 1 \}$$

$$= \{ x \in \mathbb{R} \mid x \in (2k\pi, (2k+1)\pi) \}$$



c) $f(2-x) - 1 = y$ where $f(x) = \begin{cases} 3 & x \leq 2 \\ 2-x & x > 2 \end{cases}$

Domain of $f(x-2) - 1 = \mathbb{R}$

$$f(2-x) = \begin{cases} 3 & 2-x \leq 2 \\ 2-(2-x) & 2-x > 2 \end{cases}$$

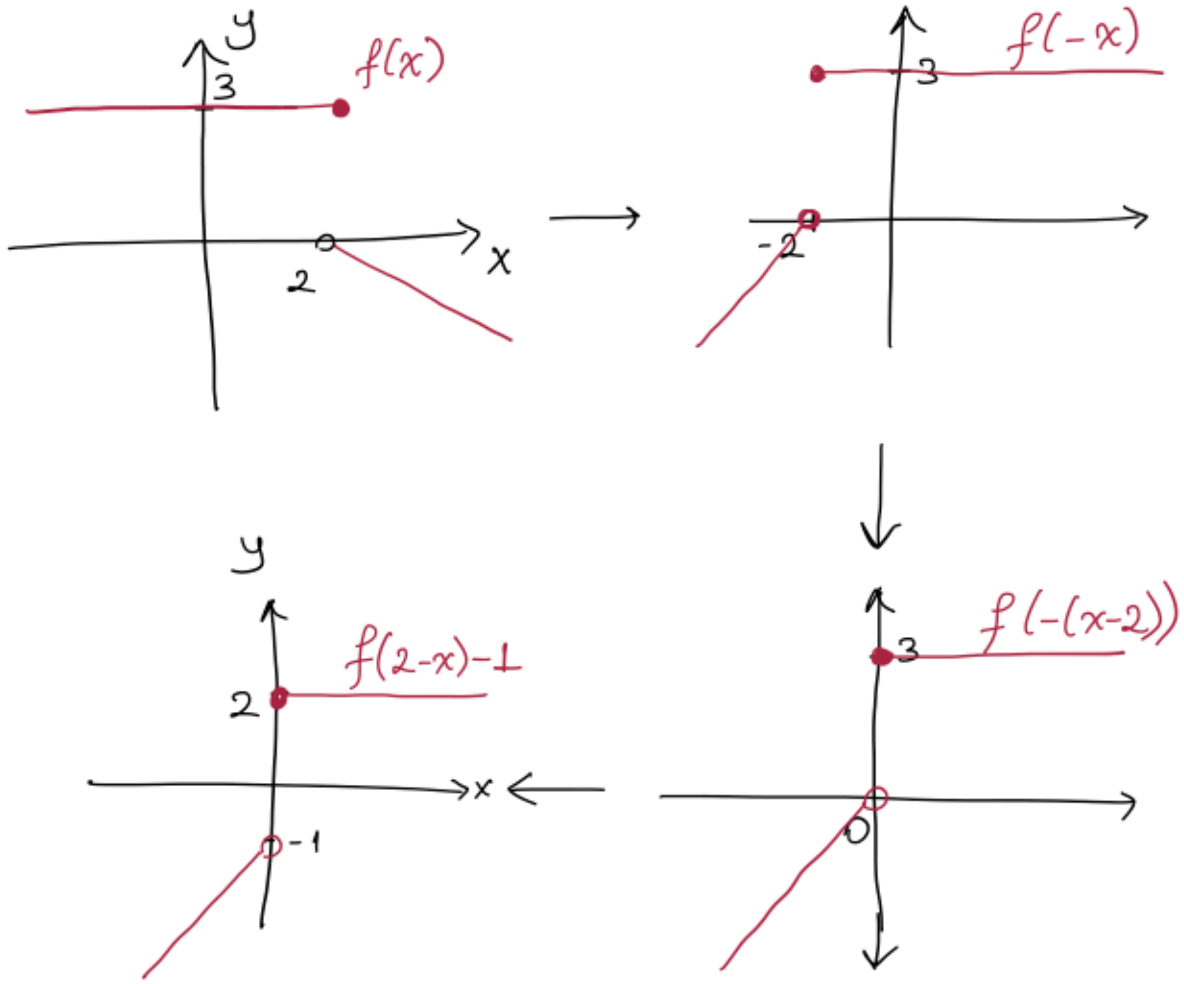
$$= \begin{cases} 3 & 0 \leq x \\ x & 0 > x \end{cases}$$

$$f(2-x) - 1 = \begin{cases} 2 & 0 \leq x \\ x-1 & 0 > x \end{cases}$$

$x-1$ is increasing function so its range on $(-\infty, 0)$ is $(-\infty, -1)$

Overall range of $f(2-x)-1$ is $(-\infty,-1) \cup \{2\}$

we will sketch the graph of $f(2-x)-1$ by using the graph of $f(x)$



d) $f(x) = 2x^2+12x+13$

↳ Exercise

Q2| Let $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0 \\ \dfrac{1}{x^2+1} & \text{if } x<0 \end{cases}$. Find $f^{-1}(x)$ if exists.

Recall: $f: A \to B$ is 1-1 & onto $\iff f^{-1}$ exists.

Domain of $f(x) = \mathbb{R}$

Range of $f(x)$: x^2+1 is increasing function on $[0,\infty)$

so its range is $[1,\infty)$

$\frac{1}{x^2+1}$ is increasing function on $(-\infty,0)$, and it is always positive so its range is $(0,1)$

Overall range of $f(x) = (0,\infty)$

$\Rightarrow f: \mathbb{R} \longrightarrow (0,\infty)$

Claim: f is 1-1 and onto.

Proof:

Since ranges of x^2+1 & $\frac{1}{x^2+1}$ are discrete which means their intersection is empty.

It is enough to check x^2+1 and $\frac{1}{x^2+1}$ are one-to-one and onto functions seperately.

- Let $x_1^2+1 = x_2^2+1 \Rightarrow x_1^2 = x_2^2 \Rightarrow |x_1| = |x_2|$

for the function x^2+1, x values are always positive

So we can say that $x_1 = x_2$ which means x^2+1 is one-to-one on $[0,\infty)$.

Exercise: Show that $\frac{1}{x^2+1}$ is one-to-one on $(-\infty,0)$

$x = \sqrt{x-1}$

• Let $x \in [1, \infty)$ then choose x_0 .
(we can make this choice since $x-1 \geq 0$ for $x \in [1,\infty)$)

then $x_0^2 + 1 = \sqrt{x-1} + 1 = x$

So the function x^2+1 is onto. on its image.

Exercise: Show that $\dfrac{1}{x^2+1}$ is onto over $(0,1)$

Overall, we showed that $f(x)$ is onto and one-to-one function which means its inverse exist.

! In order to find inverse of a function, first we need to check the existence of an inverse.

$$f^{-1}(x) = \begin{cases} \sqrt{x-1} + 1 & , x \geq 1 \\ \sqrt{\frac{1}{x} - 1} & , 1 > x \geq 0 \end{cases}$$

(Range of $f(x)$ will be the domain of $f^{-1}(x)$)

Q3| Let $f(x) = (x+1)^2$ and $g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

a) For which x, $g(x) \leq x$?

Case 1: Assume that $x \in \mathbb{Q}$.

then $g(x) = 0$
the inequality $g(x) \leq x$ holds for $x \in \mathbb{Q}$
if and only if $x \geq 0$

So the solution set for case 1 is $\mathbb{Q}^{\geq 0}$.

Case 2: Assume that $x \notin \mathbb{Q}$ then $g(x) = 1$

So the inequality holds if and only if $1 \leq x$

Solution set for case 2 is $(\mathbb{R} \setminus \mathbb{Q}) \cap [1, \infty)$

Overall solution set is $(\mathbb{Q} \cap [0,\infty)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap [1,\infty))$

b) For which x, $g(x) \leq f(x)$?

↳ Exercise

c) What is $f(g(x)) - g(x)$?

First, we will find $(f \circ g)(x)$.

If $x \in \mathbb{Q}$ then $f(g(x)) = f(0) = 1$

If $x \notin \mathbb{Q}$ then $f(g(x)) = f(1) = 4$

$$\Rightarrow f \circ g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 4 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$(f \circ g)(x) - g(x) = ?$

Again we'll examine this in cases.

If $x \in \mathbb{Q}$ then $\underbrace{(f \circ g)(x)}_{1} - \underbrace{g(x)}_{0} = 1$

If $x \notin \mathbb{Q}$ then $\underbrace{(f \circ g)(x)}_{4} - \underbrace{g(x)}_{1} = 3$

$$(f \circ g^{-1})(x) - g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 3 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Q4| Specify whether the given function is even, odd neither

a) $f(x) = \dfrac{x}{1-153^x} - \dfrac{x}{2}$

Recall: If $f(-x) = f(x)$ $\forall x \in \mathbb{R}$, then $f(x)$ is called even func.

If $f(-x) = -f(x)$ $\forall x \in \mathbb{R}$, then $f(x)$ is called odd func.

$$f(x) - f(-x) = \frac{x}{1-153^x} - \frac{x}{2} - \left(\frac{-x}{1-153^{-x}} + \frac{x}{2} \right)$$

$$= \frac{x}{1-153^x} + \frac{x \cdot 153^x}{153^x - 1} - x$$

$$= \frac{x - x153^x}{1-153^x} - x \quad = x - x$$
$$= 0$$

$\Rightarrow f(x) = f(-x)$ $\forall x$ $\Rightarrow f$ is even

b) $f(x) = x\cos x$

↳ Exercise

Q5| Prove or disprove the following

a) If f is both even and odd function, then $f(x) = 0$ $\forall x$.

TRUE

Assume that $f(x)$ is both even and odd function.

Then $f(-x)=f(x)$ and $f(-x)=-f(x)$

$$\Rightarrow f(x)=f(-x)=-f(x) \Rightarrow f(x)=-f(x) \quad \forall x$$

$$\Rightarrow f(x)=0 \quad \forall x$$

b) If g is odd function and let $h=f\circ g$, then h is odd.

FALSE

Let $g(x)=x$ and $f(x)=x^2$ then $f\circ g(x)=x^2=h(x)$

$h(x)$ is not an odd function.

c) If g is even function and let $h=f\circ g$, then h is even.

↳ Exercise

Q6| a) Show that any function $f:\mathbb{R}\to\mathbb{R}$ can be expressed as a sum of an even and an odd functions.

Let $f(x)$ be any function from $\mathbb{R}$ to $\mathbb{R}$.

$$\Rightarrow f(x) = \frac{2f(x)-f(-x)+f(-x)}{2}$$

$$= \frac{f(x)-f(-x)}{2} + \frac{f(x)+f(-x)}{2}$$

Call $f_e(x)=\dfrac{f(x)+f(-x)}{2}$ and $f_o(x)=\dfrac{f(x)-f(-x)}{2}$

Claim: $f_e(x)$ is an even function

$$f_e(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = f_e(x)$$

$2 \qquad\qquad 2$

$\Rightarrow f_e(-x) = f_e(x) \Rightarrow f_e(x)$ is an even func.

Exercise: Show that f_o is an odd function.

Overall, we expressed $f(x)$ as a sum of an even and an odd functions which are $f_e(x)$ and $f_o(x)$.

b) Show that this expression is unique.

Let $f: \mathbb{R} \to \mathbb{R}$ be any function.

Assume that $f = f_1 + f_2$ and $f = g_1 + g_2$ where f_1, g_1 are even and f_2, g_2 are odd.

$$f_1(x) + f_2(x) = g_1(x) + g_2(x)$$

$$f_1(x) - g_1(x) = g_2(x) - f_2(x)$$

Sum of two even function is even and similar result is true for odd functions too.
Thus $f_1(x) - g_1(x)$ is an even function where $g_2(x) - f_2(x)$ is an odd function.
This implies that $f_1(x) - g_1(x)$ is both even and odd function thus $f_1(x) - g_1(x) = 0$

$\Rightarrow f_1(x) = g_1(x)$

Similarly, $f_2(x) = g_2(x)$.

So this expression is unique.