Rec - 1

22 Ekim 2021 Cuma 13:33

Dilan Kangülo

email: dilan@metu.edu.tr

blog: blog.metu.edu.tr/dilan

Q1 Let $x, y \in \mathbb{R}$ such that $-3 < x < 5$ and $-2 < y < 1$

a) Find the largest and smallest possible integer values of $x^2 + 6x + 2$

$\rightarrow -3 < x < 5$ *

$x^2 + 6x + 2 = (x+3)^2 - 7$

Add 3 to each side of the inequality *

$0 < x+3 < 8$

$0 < (x+3)^2 < 64$

Substruct 7 from each sides of *

$-7 < (x+3)^2 - 7 < 57$

largest integer value for $x^2+6x+2 = 56$

smallest " " " " $= -6$

Do not directly try add two inequalities

b) Find the largest & smallest possible integer values of $xy-y$

$$xy-y=(x-1)y$$

$$-3<x<5 \Rightarrow -4<x-1<4 \quad *$$

substruct 1 from each side

$$\boxed{-2<y<1}$$

Case 1: If $y \in (-2,0)$

multiply with y; $-4y > y(x-1)$

$y(x-1) > 4y$

$$\Rightarrow 4y < y(x-1) < -4y$$

For this case $-2<y<0 \Rightarrow -8<4y<0$

$0<-4y<8$

$$-8<4y<y(x-1)<-4y<8$$

For case 1 largest $\to 7$
smallest $\to -7$

Case 2: If $y \in (0,1)$

multiply with y : • $-4y < y(x-1)$
• $y(x-1) < 4y$

$0 < y < 1 \Rightarrow 0 < 4y < 4$

$-4 < -4y < 0$

$-4 < -4y < y(x-1) < 4y < 4$

For case 2 largest $\to 3$
smallest $\to -3$

At the end, for $y \in (-2,1)$, largest value for $y(x-1)$ will be 7 & smallest one is -7.

Q2| Let $x, y \in \mathbb{R}$ such that $x < y$. Determine which of the following statements always hold.

a) $x^2 < y^2$

$x = -3$, $y = -2 \Rightarrow x^2 > y^2$

So this statement is not true for all $x < y \in \mathbb{R}$.

If $x=-3$, $y=-2$ then $x=-3<-2=y$ holds
so this example is countroexemple.

b) $\frac{1}{x} > \frac{1}{y}$

! If you are given an example which shows that the given statement is not true then this example is called counterexample.

$x=-1$, $y=1$ $\Rightarrow$ $x=-1<1=y$ ✓

$$\frac{1}{x} = -1 < 1 = \frac{1}{y} \Rightarrow \frac{1}{y} > \frac{1}{x}$$

This is a counterexample so this statement is not true for all $x<y \in \mathbb{R}$.

c) $x^3 < y^3$

TRUE

Case 1: Assume that $xy > 0$

$x<y \leftarrow$ known

what we know also? $\rightarrow x^2 > 0$
$y^2 > 0$

multiply with x^2 $\rightarrow$ $x^2 x < y x^2 \Rightarrow x^3 < x^2 y$ ✓

multiply with xy $\rightarrow$ $xy\, x < y\, xy \Rightarrow x^2y < xy^2$

multiply with y^2 $\rightarrow$ $y^2 x < y\, y^2 \Rightarrow xy^2 < y^3$

$\Rightarrow x^3 < x^2y < xy^2 < y^3 \Rightarrow x^3 < y^3$ //

We showed that for $xy > 0$ we have $x^3 < y^3$

Case 2: Assume that $xy < 0$

↳ Exercise

$\Rightarrow$ For this case, $x^3 < y^3$

$\Rightarrow$ At the end the statement $x^3 < y^3$ is true for $x, y \in \mathbb{R}$ s.t $x < y$.

d) $xy < y^3$

↳ Exercise

Hint: it is false

Q3| Find the values of $x \in \mathbb{R}$ such that

$$|x+2| + |x-1| = 153$$

Let $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \end{cases}$

$\{ -x$ if $x<0$

$x\in(-\infty,-2)$ $\quad -2$	$x\in[-2,1]$ $\quad 1$	$x\in(1,\infty)$
$-x-2-x+1=153$	$x+2-x+1=153$	$x+2+x-1=153$
$-2x-1=153$	$3=153$	$2x=152$
$-154=2x$	No solution	$x=76$
$-77=x$		$76\in(1,\infty)$
$-77\in(-\infty,-2)$		$\downarrow$
$\downarrow$		Sol^n for this case $\{76\}$
Sol^n for this case $\{-77\}$		

$\Rightarrow$ Solution set $\{-77, 76\}$

Q4| Solve the following inequalities

a) $|x+3|-5\geq x$

If $x\in(-\infty,-3)$

$|x+3|=-x-3$

$\Rightarrow -x-3-5\geq x$

$-8\geq 2x$

$-4\geq x$

$\Rightarrow x\in(-\infty,-4]$

If $x\in[-3,\infty)$

$|x+3|=x+3$

$\cancel{x}+3-5\geq\cancel{x}$

$-2\geq 0$ ↯

contradiction

There is no solution for this case

$\Rightarrow x \in (-\infty, -3) \cup (-\infty, -4]$

$$= (-\infty, -4]$$

Sol^n for first case

At the end, we have solution set $(-\infty, -4]$

b) $|x^2 - 7| < 2$

$|x^2 - 7| = |(x - \sqrt{7})(x + \sqrt{7})| < 2$

Case 1: $x \in (-\infty, -\sqrt{7}]$

$|\underbrace{(x - \sqrt{7})}_{<0}\underbrace{(x + \sqrt{7})}_{<0}| = (x - \sqrt{7})(x + \sqrt{7}) = x^2 - 7$

$(x - \sqrt{7})(x + \sqrt{7}) < 2$

$x^2 < 9$

$-3 < x < 3 \Rightarrow x \in (-3, 3)$

Sol^n set: $x \in (-3, 3) \cap (-\infty, -\sqrt{7}]$

$\in (-3, -\sqrt{7}]$

Case 2: $x \in (-\sqrt{7}, \sqrt{7})$

$|\underbrace{(x - \sqrt{7})}_{<0}\underbrace{(x + \sqrt{7})}_{>0}| = 7 - x^2 < 2$

$\Rightarrow 5 < x^2$

$$\Rightarrow \quad x > \sqrt{5} \text{ or } x < -\sqrt{5}$$

$$\Rightarrow \quad x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$$

Soln set: $x \in \big((-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)\big) \cap (-\sqrt{7}, \sqrt{7})$

$$\in (-\sqrt{7}, -\sqrt{5}) \cup (\sqrt{5}, \sqrt{7})$$

Case 3: $x \in [\sqrt{7}, \infty)$

↳ Exercise

c) $\dfrac{1}{|x-2|} \geq \dfrac{3}{4}$

Case 1: $x \in (-\infty, 2)$

$|x-2| = 2-x$

$$\frac{1}{2-x} \geq \frac{3}{4} \Rightarrow 4 \geq 3(2-x)$$

since $2-x > 0$

$$\Rightarrow 4 \geq 6 - 3x$$

$$3x \geq 2 \Rightarrow x \geq \frac{2}{3}$$

Soln set $\Rightarrow x \in \left[\frac{2}{3}, \infty\right) \cap (-\infty, 2)$

$$\in \left[\frac{2}{3}, 2\right)$$

Case 2: $x \in (2, \infty)$

$$|x-2| = x-2$$

$$\frac{1}{x-2} \geq \frac{3}{4} \xRightarrow[\substack{\text{since}\\ x-2>0}]{} 4 \geq 3x-6$$

$$\frac{10}{3} \geq x$$

$$\text{Sol}^n \text{ set} = x \in \left(-\infty, \tfrac{10}{3}\right] \cap (2,\infty)$$

$$\in \left(2, \tfrac{10}{3}\right]$$

At the end Soln set $= \left[\frac{2}{3}, 2\right) \cup \left(2, \frac{10}{3}\right]$

$$= \left[\tfrac{2}{3}, \tfrac{10}{3}\right] - \{2\}$$

Q5| Find the interval of $x \in \mathbb{R}$ such that $x < x^2 < |x|$

Case 1: $x \in (0,\infty) \Rightarrow |x| = x$

$$x < x^2 < x \xRightarrow[\substack{\text{cancel}\\ x \text{ since}\\ x>0}]{} 1 < x < 1 \Rightarrow \text{No solution}$$

Case 2: $x = 0$

$$\underset{x}{0} < \underset{x^2}{0} < \underset{|x|}{0} \Rightarrow \text{No sol}^n$$

Case 3: $x \in (-\infty, 0) \Rightarrow |x| = \ldots$

Case 3: $x \in (-\infty, 0) \Rightarrow |x| = -x$

$x < x^2 < -x$

- $x < x^2 \Longrightarrow 1 > x$

cancel x but be careful since $x < 0$

- $x^2 < -x \Longrightarrow x > -1$

Sol^n set $= x \in (-\infty, 0) \cap (-1, \infty) \cap (-\infty, 1)$
$\in (-1, 0)$

$(-\infty, 0) \subset (-\infty, 1)$

Q6| Find the domain and range of each function and sketch their graphs (use shifting and scaling)

a) $f(x) = \sqrt{x^2 - 1}$

Domain of $f(x) = \mathbb{R} - (-1, 1)$

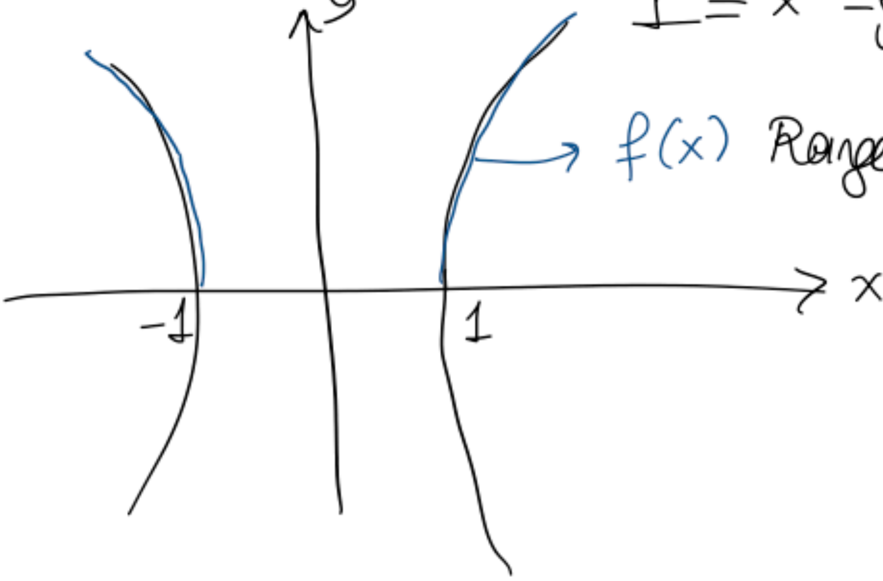
$x^2 - 1 \geq 0 \Rightarrow x^2 \geq 1 \Rightarrow x \geq 1$ or $x \leq -1$
$\Rightarrow x \in (-\infty, -1] \cup [1, \infty)$
$\Rightarrow x \in \mathbb{R} - (-1, 1)$

$y = \sqrt{x^2 - 1}$

$\rightarrow y = x^2 - 1$

$1 = x^2 - y^2 \leftarrow$ hyperbola

$f(x)$ Range $f(x) = [0, \infty)$



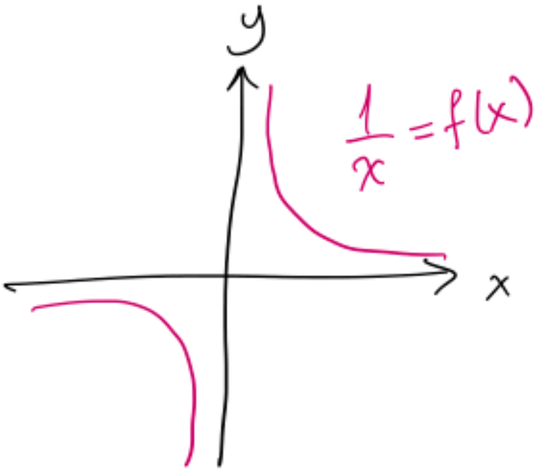
b) $f(x) = \dfrac{2-x}{x-1}$

Domain of $f(x) = \mathbb{R} - \{1\}$

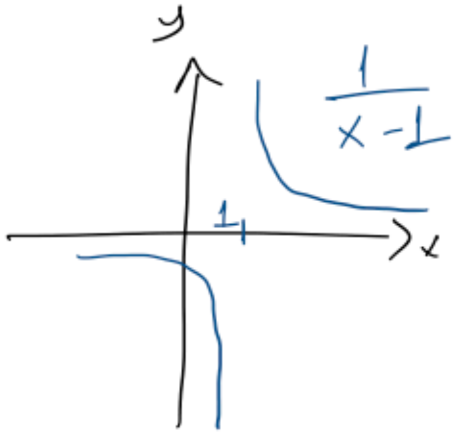
$$f(x) = \frac{2-x}{x-1} = (-1)\cdot\frac{x-1-1}{x-1}$$

$$= -1 + \frac{1}{x-1}$$

Range of $f(x) = \mathbb{R} - \{-1\}$



$\Longrightarrow$



$\Downarrow$ $\quad -1 + \frac{}{x-1}$