Week 15

24 Haziran 2021 Perşembe 08:40

Q1] Evaluate $\oint_C (3x+4y)\,dx + (2x+3y^2)\,dy$ where C is the curve $x^2+y^2=4$ oriented counter clockwise

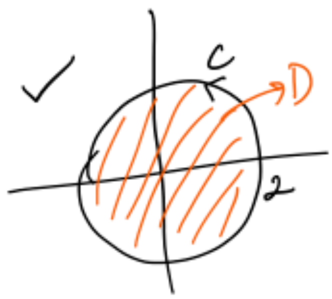
Green's theorem:

Let C be counter-clockwisely oriented, simple, piece-wise smooth closed curve in the plane and let D be the simply-connected region bounded by C. If P & Q have continuous partial derivatives on an open region contains D, then

$$\oint_C F\cdot dr = \iint_D (Q_x - P_y)\, dA$$

$P = (3x+4y)$, $Q = 2x+3y^2$

C is simple, ccw, piecewise-smooth, closed ✓



D is simply connected ✓

P & Q have cont. partial derivatives

So we can apply Green's theorem.

$$\oint_C \underbrace{(3x+4y)}_{P}\,dx + \underbrace{(2x+3y^2)}_{Q}\,dy = \iint_D (2-4)\,dA = -2 \cdot \underbrace{\iint_D 1\,dA}_{\text{area of } D}$$

$$= -2 \cdot \pi 4$$

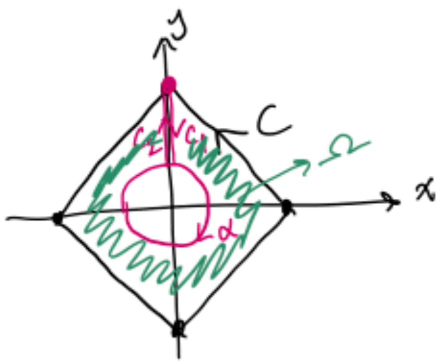
$$= -8\pi$$

Q2] Given the vector field

2a) Given the ...

$$F = \frac{y\vec{i} - x\vec{j}}{x^2+y^2}$$

compute $\oint_C F\,dr$ where C is the curve enclosing the square with vertices $(2,0), (0,2), (-2,0), (0,-2)$ oriented in the counter clockwise direction.



The region D contains $(0,0)$ and we know that P, Q does not have cont. partial derivative at that point $\Rightarrow$ We cannot apply Green's theorem since the region enclosed by C & P, Q have cont partial derivatives is $D - \{(0,0)\}$ which is not simply-connected.

$C_1 : (0, y)$, $y=2 \rightarrow y=1$

$C_2 : (0, y)$, $y=1 \Rightarrow y=2$

Let $\gamma = C \cup C_1 \cup \alpha \cup C_2 \Rightarrow \gamma$ is closed, simple, piecewise smooth

$$P = \frac{y}{x^2+y^2}, \quad Q = \frac{-x}{x^2+y^2}$$

$$P_y = \frac{x^2+y^2 - y(2y)}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}, \quad Q_x = \frac{-(x^2+y^2)-(-x)\cdot(2x)}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Since Ω is simply-connected, F is conservative on Ω

$$\oint_\gamma F\,dr = 0 \quad \text{since } F \text{ is conservative on } \Omega.$$

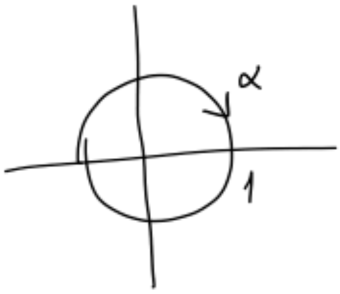
$$0 = \oint_{C \cup C_1 \cup \alpha \cup C_2} F\,dr = \oint_C F\,dr + \oint_{C_1} F\,dr + \oint_\alpha F\,dr + \oint_{C_2} F\,dr$$

Observe that $C_1 = -C_2$

$$0 = \oint_C F\,dr - \cancel{\oint_{C_2} F\,dr} + \oint_\alpha F\,dr + \cancel{\oint_{C_2} F\,dr}$$

$$\Rightarrow \oint_C F\,dr = -\oint_\alpha F\,dr$$

$\alpha \rightarrow$ unit circle oriented cw



$$\left.\begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array}\right\} \theta \in [0, 2\pi] \rightarrow \text{This parametrization is ccw}$$

$$-\alpha : (x, y) = (\cos\theta, \sin\theta),\ \theta \in [0, 2\pi]$$

$$dx = -\sin\theta\, d\theta,\quad dy = \cos\theta\, d\theta$$

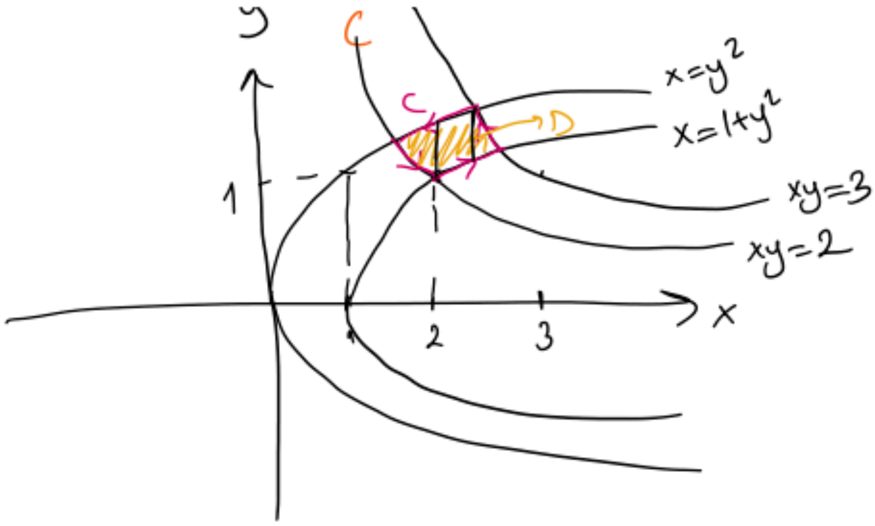
$$\oint_C F\,dr = -\left(-\oint_{-\alpha} F\,dr\right) = \oint_{-\alpha} F\,dr$$

$$= \int_0^{2\pi} \left(\frac{\sin\theta}{\underbrace{\cos^2\theta + \sin^2\theta}_{1}} (-\sin\theta) - \frac{\cos\theta}{\underbrace{\cos^2\theta + \sin^2\theta}_{1}} \cos\theta \right) d\theta$$

$$= \int_0^{2\pi} -\sin^2\theta - \cos^2\theta \; d\theta = -2\pi$$

Q3) Let C be the boundary of the region bounded by $x = y^2$, $x = 1 + y^2$, $xy = 2$, $xy = 3$ which is oriented ccw.

Evaluate $\displaystyle \oint \left(e^{x^3} - \frac{2}{3} y^3\right) dx + \left(\sin y^3 + \frac{1}{2} x^2\right) dy$



P, Q have cont. partial derivatives on D.
D is simply-connected
C is precewise-smooth, simple, closed, ccw
We can apply Green's Theorem.

$$\oint_F \underbrace{\left(e^{x^3}-\frac{2}{3}y^3\right)}_{P} dx + \underbrace{\left(\sin y^3+\frac{1}{2}x^2\right)}_{Q} dy = \iint_D x+2y^2\, dA$$

Since our region need to be divided into 3 part in the iteration, we can use change of variable.

$$u = y^2 - x \quad \Rightarrow \quad -1 \le u \le 0$$

$$v = xy \quad \Rightarrow \quad 2 \le v \le 3$$

$$\iint_{D_{xy}} f(x,y)\, dA_{xy} = \iint_{D_{u,v}} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{u,v}$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} = \frac{1}{\begin{vmatrix} -1 & 2y \\ y & x \end{vmatrix}}$$

$$= \frac{1}{|-x-2y^2|} = \frac{1}{|g(u,v)|}$$

Call, $g(u,v) = x+2y^2$

$$\iint \ldots \iint \ldots \cancel{\ } \frac{1}{\ } \, dA_{u,v} = \iint^{3\ 0} 1\, du\, dv$$

$$\iint_D x+2y^2 \, dA = \iint_{D_{uv}} \cancel{g(u,v)} \cdot \underbrace{\left|\cancel{g(u,v)}\right|}_{>0} \qquad \begin{matrix} \\ 2 \ -1 \end{matrix}$$

$\forall u \in [-1,0]$
$v \in [2,3]$ $\quad = 1 \,//$

Q4| Find the area enclosed by the curve which is oriented counterclockwise $\vec{r}(t) = \left(\dfrac{2\cos t - \sin t}{2}, \sin t\right) \quad t \in [0, 2\pi]$

Hint: Apply Green's theorem for the vector field $F = \left(-\dfrac{y}{2}, \dfrac{x}{2}\right)$ along the curve $r(t)$.

$\iint_D 1 \, dA \rightarrow$ area of the region D enclosed by $r(t)$

$\Rightarrow$ Then $Q_x - P_y = 1$

In the suggestion we have $F = (\overset{P}{\overline{-\tfrac{y}{2}}}, \overset{Q}{\widehat{\tfrac{x}{2}}}) \Rightarrow Q_x = \frac{1}{2}$

$P_y = -\frac{1}{2}$

$\Rightarrow \boxed{Q_x - P_y = 1} \Leftarrow$

Observe that $r(t)$ is closed, simple, piecewise-smooth & P,Q have cont. partial derivatives on D which is simply connected. Therefore we can apply Green's Thm

$$\oint_{r(t)} F \, dr = \iint_D 1 \, dA \rightarrow \text{area of region enclosed by } r(t).$$

$$\oint_{r(t)} \frac{-y}{2} \, dx + \frac{x}{2} \, dy \,, \qquad r(t) = \left(\cos t - \frac{\sin t}{2}, \sin t\right) \quad t \in [0, 2\pi]$$

$x = \cos t - \dfrac{\sin t}{2}$

2π $\quad ||$

$$\frac{1}{2}\int_0 \left(-\sin t\left(-\sin t-\frac{\cos t}{2}\right)+\left(\cos t-\frac{\sin t}{2}\right)\cos t\right)dt$$

$dx = \left(-\sin t - \frac{\cos t}{2}\right)dt$

$y = \sin t$

$dy = \cos t \, dt$

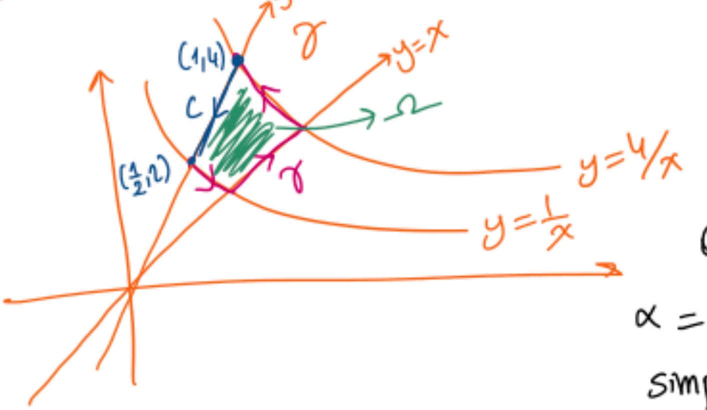
$$= \frac{1}{2}\int_0^{2\pi} \underbrace{\sin^2 t + \cancel{\frac{\sin t\cos t}{2}} + \cos^2 t}_{1} - \cancel{\frac{\sin t\cos t}{2}} \, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 1 \, dt = \pi$$

As an exercise, try to use $F = (y, 2x)$

$P_y = 1$, $Q_x = 2 \Rightarrow Q_x - P_y = 1$

Q5) Evaluate $\int_\gamma -\frac{y^2}{x}\,dx + y\ln x \, dy$ along the curve γ



Let C be the line segment from $(1,4)$ to $(\frac{1}{2}, 2)$

Observe that the curve $\alpha = \gamma \cup C$ is closed, piecewise-smooth simple, ccw

P, Q have cont. partial derivative on Ω which simply connected.

Therefore we can apply Green's Theorem to $\oint_\alpha F\,dr$.

$$\oint_\alpha F\,dr = \iint_\Omega (Q_x - P_y)\,dA$$

$$= \iint \frac{y}{x} - \left(-\frac{2y}{x}\right) dA = \iint \frac{3y}{x}\,dA$$

$$\iint_\Omega \frac{\ }{x} \quad \cdots x) \qquad \iint_\Omega x$$

To evaluate this double integral, we'll use change of variable

$u = \frac{y}{x}$, $v = xy$, $1 \le u \le 4$, $1 \le v \le 4$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix}} = \frac{1}{\left|-\frac{2y}{x}\right|} = \frac{1}{|-2u|}$$

$$\iint_\Omega \frac{3y}{x}\, dA = \iint_{\Omega_{u,v}} 3u \left|\frac{-1}{2u}\right| \cdot dA_{u,v} = \int_1^4\int_1^4 3u \cdot \left|-\frac{1}{2u}\right| du\,dv$$

$$= 24$$

$$\oint_\alpha F\, dr = 24 = \oint_{\gamma \cup C} F\,dr = \oint_\gamma F\,dr + \oint_C F\,dr$$

C is on the line $y = 4x$, $(1,4) \to (\frac{1}{2}, 2)$

$$\oint_\gamma F\,dr = 24 - \oint_C F\,dr = 24 + \oint_{-C} F\,dr$$

parametrization of $-C$: $(t, 4t)$, $t \in [\frac{1}{2}, 1]$

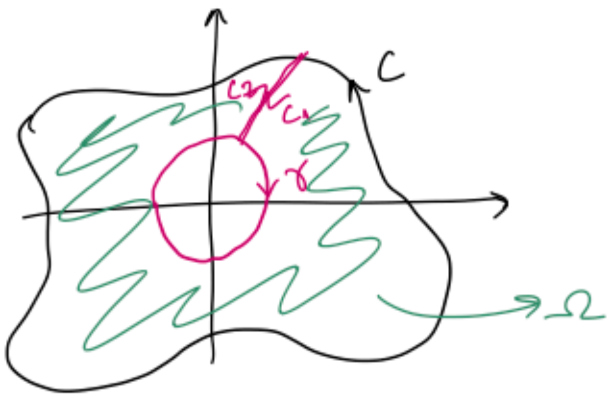
$$\oint_\gamma F\,dr = 24 + \int_{1/2}^{1} -\frac{4t^2}{t} + 4t\ln(t)\cdot 4 \quad dt$$

$\Rightarrow$ rest is an exercise.

Q6| Consider $I(C) = \oint_C \frac{(x-y)}{x^2+y^2}dx + \frac{(x+y)}{x^2+y^2}dy$

where C is an arbitrary simple, closed, piecewise-smooth ccw oriented curve

a) Evaluate $I(C)$ when $(0,0)$ is inside C.



Assume that C contains unit circle in it.

$\alpha = C \cup C_1 \cup \gamma \cup C_2$ will be closed curve

$$P = \frac{x-y}{x^2+y^2} \Rightarrow P_y = \frac{-(x^2+y^2)-(x-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-2xy+y^2}{(x^2+y^2)^2}$$

$$Q = \frac{x+y}{x^2+y^2} \Rightarrow Q_x = \frac{(x^2+y^2)-(x+y)2x}{(x^2+y^2)^2} = \frac{-x^2-2xy+y^2}{(x^2+y^2)^2}$$

For $(x,y) \neq (0,0)$, F is conservative

Since Ω does not contain $(0,0)$, $\oint_{C \cup C_1 \cup \gamma \cup C_2 = \alpha} F\,dr = 0$

We know that $C_1 = -C_2$

$\oint_C F\,dr = -\oint_\gamma F\,dr$ where γ is unit circle oriented cw.

$$= \oint F\,dr = \int^{2\pi} (\cos\theta - \sin\theta)(-\sin\theta) + (\cos\theta + \sin\theta)\cos\theta\, d\theta$$

$$\underset{\underset{-\gamma}{\cup}}{\left.\begin{array}{l} x=\cos\theta \\ y=\sin\theta \end{array}\right\}} \theta\in[0,2\pi] \quad \underset{\underset{0}{\cup}}{=} \int_0^{2\pi} -\cancel{\sin\theta\cos\theta}+\underbrace{\sin^2\theta + \cos^2\theta}_{1}+\cancel{\sin\theta\cos\theta}\, d\theta$$

$$= 2\pi \ //$$

If C does not contain unit circle then it would contain some circle with radius ε, for some $\varepsilon>0$.

when we have $x=\varepsilon\cos\theta$ $\theta\in[0,2\pi]$
$y=\varepsilon\sin\theta$

$$\int_0^{2\pi} \frac{-\varepsilon^2\sin\theta\cos\theta+\varepsilon^2\sin\theta+\varepsilon^2\cos\theta+\varepsilon^2\sin\theta\cos\theta}{\varepsilon^2}\, d\theta = 2\pi //$$

b) If $(0,0)$ is outside of C.

Then the curve C & enclosed region D satisfies the conditions for Green's theorem

Since F is conservative $Q_x - P_y = 0 \Rightarrow \oint_C F\,dr = 0$