Week 14

17 Haziran 2021 Perşembe 08:42

Q1) Determine whether the given vector field is conservative find a potential if it is.

a) $F(x,y,z) = e^{x^2+y^2+z^2}(xz\vec{i} + yz\vec{j} + xy\vec{k})$

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$ is conservative field if there exists $\phi: \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{R}^3 \to \mathbb{R}$ so that $\nabla\phi = F$

Theorem: If $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$ is conservative field where P, Q, R has continuous first-order partial derivatives on the domain D, then $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ (*)

Remark: If one of these equations does not satisfied, then we can say F is not conservative.

If all of these equations satisfied, then we can not conclude anything.

Theorem: Let domain D be simply-connected. F is conservative field on D if and only if (*) satisfied.

$F = P\vec{i} + Q\vec{j} + R\vec{k}$ then $P = e^{x^2+y^2+z^2}xz$

$Q = e^{x^2+y^2+z^2}yz$

$R = e^{x^2+y^2+z^2}xy$

$\frac{\partial P}{\partial y} = e^{x^2+y^2+z^2} \cdot 2yxz$

$\parallel$ ✓

$\frac{\partial Q}{\partial x} = e^{x^2+y^2+z^2} \cdot 2xyz$

$\frac{\partial P}{\partial z} = e^{x^2+y^2+z^2} \cdot 2xz^2 + e^{x^2+y^2+z^2} \cdot x$

$\nparallel$

$\frac{\partial R}{\partial x} = e^{x^2+y^2+z^2} \, 2x^2y + e^{x^2+y^2+z^2} \cdot y$

$\Rightarrow F$ is $\underline{\underline{\text{not}}}$ conservative.

b) $F(x,y) = (y\cos x - \cos y)\vec{i} + (\sin x + x\sin y)\vec{j}$.

$$\frac{\partial P}{\partial y} \overset{?}{=} \frac{\partial Q}{\partial x} \Rightarrow \frac{\partial P}{\partial y} = \cos x + \sin y$$

$$\frac{\partial Q}{\partial x} = \cos x + \sin y$$

Since $\mathbb{R}^2$ is simply-connected domain & $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

we can say that F is conservative.

Let ϕ be a potential for F, then $\nabla\phi = F$

$$\nabla\phi = (\phi_x, \phi_y)$$

$$(\phi_x, \phi_y) = (P, Q)$$

$$\phi(x,y) = \int \phi_x(x,y)\,dx = \int P\,dx$$

$$= \int y\cos x - \cos y\,dx$$

$$= y\sin x - x\cos y + \underline{\underline{c(y)}}$$ → a function depend on y

$$\frac{\partial}{\partial y}\phi(x,y) = \frac{\partial}{\partial y}(y\sin x - x\cos y + c(y))$$

$$= \sin x + x\sin y + c'(y) = Q = \sin x + x\sin y$$

$$\Rightarrow c'(y) = 0 \Rightarrow c(y) = c_1 \in \mathbb{R}$$

$$\Rightarrow \phi(x,y) = y\sin x - x\cos y + c_1, \quad c_1 \in \mathbb{R}.$$

$\Rightarrow \phi$ is a potential for F.

Q21 (a) Find a potential function for the vector field

$$F(x,y) = \underbrace{2xy}_{P}\,\vec{i} + \underbrace{(x^2+y^2)}_{Q}\,\vec{j}$$

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x} = 2x \quad , \ \mathbb{R}^2 \text{ is simply-connected domain}$$

So F has potential on $\mathbb{R}^2$.

$$\nabla\phi = F \Rightarrow (\phi_x, \phi_y) = (P, Q)$$

$$\phi(x,y) = \int \phi_x(x,y)\,dx = \int P\,dx = \int 2xy\,dx$$

$$= x^2 y + c(y)$$

$$\left.\begin{aligned} \phi_y(x,y) &= x^2 + c'(y) \\ &\parallel \\ Q &= x^2 + y^2 \end{aligned}\right\} \quad \begin{aligned} c'(y) &= y^2 \\ c(y) &= \frac{y^3}{3} + c_1 \end{aligned}$$

$$\phi(x,y) = x^2 y + \frac{y^3}{3} + c_1, \ c_1 \in \mathbb{R} \leftarrow \text{potential function for } F.$$

(b) For a vector field in part (a), compute the line integral on C, where C is the portion of the curve $\sqrt{x} + xy + \sqrt{y} = 7$ starting at $(4,1)$ and ending at $(1,4)$

$$\int_C F \cdot dr$$

Theorem: If D is simply-connected domain, and F is smooth vector field TFAE

i) F is conservative

ii) $\int_C F\,dr$ is path independent.

$$\parallel$$

$$\phi(b) - \phi(a) \quad \text{where } \nabla\phi = F$$

Since F is conservative, $\int F\,dr = \phi(1,4) - \phi(4,1)$

$$\check{C}$$

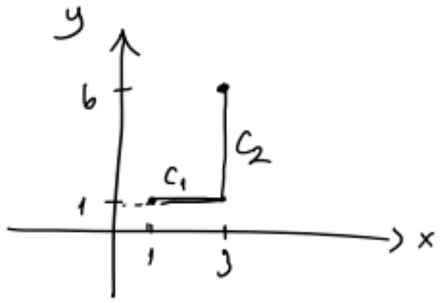
$$= 4 + \frac{64}{3} + C_1 - 16 - \frac{1}{3} - C_1$$

$$= \underline{\underline{9}}$$

Q3| Evaluate $\int_C \frac{1}{xy}\,dx + \frac{1}{x+y}\,dy$ along the path from $(1,1)$ to $(3,1)$ to $(3,6)$ using straight line segments.

$$\int_C F\,dr = \int_C \frac{1}{xy}\,dx + \frac{1}{x+y}\,dy \Rightarrow F = (P,Q) = \left(\frac{1}{xy}, \frac{1}{x+y}\right)$$

$$\frac{\partial P}{\partial y} = -\frac{1}{xy^2} \neq \frac{\partial Q}{\partial x} = -\frac{1}{(x+y)^2} \Rightarrow F \text{ is } \underline{\underline{\text{not}}} \text{ conservative.}$$



$$C = C_1 \cup C_2$$

$$\int_C F\,dr = \int_{C_1 \cup C_2} F\,dr = \int_{C_1} F\,dr + \int_{C_2} F\,dr$$

since $C_1 \cap C_2 = \{(3,1)\}$

$\underline{C_1}$: $1 \le x \le 3$, $y = 1$ $\Big\}$ $x = t \Downarrow C_1 : (t,1)$, $t \in [1,3]$

$dx = dt$, $dy = 0$

$$\int_{C_1} \frac{1}{xy}\,dx + \cancel{\frac{1}{x+y}\,dy}^{\,0} = \int_1^3 \frac{1}{t \cdot 1}\,dt = \ln|3| - \ln|1|$$

$$= \ln|3|$$

$\underline{C_2}$: $x = 3$, $1 \le y \le 6$ $\Big\}$ $y = t \Downarrow C_2 : (3,t)$, $t \in [1,6]$

$dx = 0$, $dy = dt$

$dx = 0 \quad , \quad dy = dt$

$$\int_{C_2} \frac{1}{xy}\,\underbrace{\cancel{dx}}_{0} + \frac{1}{x+y}\,dy = \int_1^6 \frac{1}{3+t}\,dt = \ln|3+t|\Big|_1^6$$

$$= \ln|9| - \ln|4|$$

$$= \ln\left|\frac{9}{4}\right|$$

$$\int_C F\,dr = \ln|3| + \ln\left|\frac{9}{4}\right| = \ln\left|\frac{27}{4}\right|$$

Q4) The field $F(x,y,z) = \underbrace{(axy+z)}_{P}\vec{i} + \underbrace{x^2}_{Q}\vec{j} + \underbrace{(bx+2z)}_{R}\vec{k}$ is conservative.

a) Find a & b and a potential for F.

Since F is conservative, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

$ax = 2x \quad , \quad 1 = b \quad , \quad 0 = 0 \quad \Rightarrow \quad a = 2 \ \& \ b = 1$

$F(x,y,z) = (2xy+z,\ x^2,\ x+2z) = (P,Q,R)$

We are given that F is conservative, so there exists

$\phi : \mathbb{R}^3 \to \mathbb{R}$ so that $\nabla\phi = F$

$\Rightarrow (\phi_x, \phi_y, \phi_z) = (P, Q, R)$

$$\phi(x,y,z) = \int \phi_x(x,y,z)\,dx = \int P\,dx = \int 2xy+z\,dx$$

$$= x^2y + xz + c(y,z)$$

$$\frac{\partial}{\partial y}\phi(x,y,z) = \frac{\partial}{\partial y}\left(x^2y + xz + c(y,z)\right)$$

$$= x^2 + c_y(y,z) = Q = x^2$$

$\Rightarrow c_y(y,z) = 0 \Rightarrow c(y,z)$ is constant on y

therefore it is a function dependent on z only.

$$\Rightarrow c(y,z) = c(z)$$

$$\Rightarrow \phi(x,y,z) = x^2y + xz + c(z)$$

$$\frac{\partial}{\partial z}\phi(x,y,z) = \frac{\partial}{\partial z}\left(x^2y + xz + c(z)\right)$$

$$= x + c'(z) = R = x + 2z$$

$$\Rightarrow c'(z) = 2z \Rightarrow c(z) = z^2 + c_1, \quad c_1 \in \mathbb{R}$$

A potential for F is $\phi(x,y,z) = x^2y + xz + z^2 + c_1, \quad c_1 \in \mathbb{R}$

b) Evaluate $\int_C F.dr$ where C is the curve from $(1,1,0)$ to $(0,0,3)$ that lies in the intersection of the surfaces $2x+y+z=3$, $9x^2+9y^2+2z^2=18$ in the first octant., F is the same in part a.

Since F is conservative, $\int_C F dr$ is independent of path.

$$\int_C F dr = \phi(0,0,3) - \phi(1,1,0), \quad \nabla\phi = F.$$

$$= 9 + c_1 - 1 - c_1 = \underline{\underline{8}}$$

Q5| Evaluate the line integral

$$\int_C \underbrace{(2x\sin(\pi y) - e^z)}_{P}\,dx + \underbrace{(\pi x^2\cos(\pi y) - 3e^z)}_{Q}\,dy \underbrace{- xe^z}_{R}\,dz$$

where C is the intersection of the surfaces $z = \ln(1+x)$, $y = x$

from the point $(0,0,0)$ to $(1,1,\ln(2))$

Let call $F=(P,Q,R)$

$$\frac{\partial P}{\partial y} = 2x\pi\cos(\pi y) = \frac{\partial Q}{\partial x} = 2\pi x\cos(\pi y) \checkmark$$

$$\frac{\partial P}{\partial z} = -e^z = \frac{\partial R}{\partial x} = -e^z \checkmark$$

$$\frac{\partial Q}{\partial z} = -3e^z \neq \underbrace{\frac{\partial R}{\partial y} = 0}_{} \times \quad \longrightarrow \frac{\partial R}{\partial y} = -3e^z$$

$\Rightarrow F$ is <u>not</u> conservative.

Let $F^*(x,y,z) = (P, Q, \underbrace{R-3ye^z}_{R'})$

$$\frac{\partial R'}{\partial y} = -3e^z = \frac{\partial Q}{\partial z}$$ others will be the same

$\Rightarrow F^*$ is conservative since $\mathbb{R}^3$ is simply-connected domain.

$\int_C F^* dr \to$ is independent of path

$$\underbrace{\int_C F^* dr}_{} = \int_C F\, dr + \int_C -3ye^z\, dr$$

Let's try to find potential for F^*

$\nabla\phi = F$, $(\phi_x, \phi_y, \phi_z) = (P, Q, R') = (P, Q, R-3ye^z)$

$$\phi(x,y,z) = \int \phi_x\, dx = \int P\, dx = \int 2x\sin(\pi y) - e^z\, dx$$

$$= \underline{\underline{x^2\sin(\pi y) - xe^z + c(y,z)}}$$

$$\left.\begin{aligned} \phi_y(x,y,z) &= \pi x^2\cos(\pi y) + c_y(y,z) \\ \overset{\parallel}{Q} &= \pi x^2\cos(\pi y) - 3e^z \end{aligned}\right\} \quad \begin{aligned} c_y(y,z) &= -3e^z \\ \underline{\underline{c(y,z)}} &= -3e^z y + d(z) \end{aligned}$$

$$\phi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z + d(z)$$

$$\left.\begin{aligned} \phi_z(x,y,z) &= -xe^z - 3ye^z + d'(z) \\ \overset{\parallel}{R'} &= R - 3ye^z = -xe^z - 3ye^z \end{aligned}\right\} \quad \begin{aligned} d'(z) &= 0 \\ d(z) &= c_1 \,,\ c_1 \in \mathbb{R} \end{aligned}$$

A potential for F^*, $\phi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z + c_1$, $c_1 \in \mathbb{R}$

$$\int_C F^* dr = \phi(1,1,\ln 2) - \phi(0,0,0)$$

$$= -2 - 3.2 + c_1 - 0 - c_1 = -8$$

$$\int_C F\,dr = \underbrace{\int_C F^* dr}_{-8} + \underline{\underline{\int_C 3ye^z\,dz}} \qquad \left(F = F^* + 3ye^z \vec{k} \right)$$

$\underline{C:}$ $z = \ln(1+x) \cap y = x$, $(0,0,0) \to (1,1,\ln 2)$

$x = t,\ y = t,\ z = \ln(1+t)$, $t \in [0,1]$

$dx = dt$, $dy = dt$, $dz = \frac{1}{1+t}\,dt$

$$\int_C 3y\underline{\underline{e^z}}\,dz = \int_0^1 3 \cdot t \cdot e^{\ln(1+t)} \cdot \frac{1}{1+t}\,dt$$

$$= \int_0^1 \frac{3t\,(1+t)}{1+t}\,dt = \int_0^1 3t\,dt = \frac{3t^2}{2}\Big|_0^1$$

$$= \frac{3}{2}$$

$$\int_C F\,dr = -8 + \frac{3}{2} = -\frac{13}{2} \ //$$