Week 13

10 Haziran 2021 Perşembe 08:33

Q1 Parametrize the curve of intersection of the given surfaces.

a) The elliptic cylinder $\frac{x^2}{9}+\frac{y^2}{4}=1$ with the surface $z=xy$

$x=3\cos t$, $t\in[0,2\pi)$
$y=2\sin t$

$\Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=\frac{9\cos^2 t}{9}+\frac{4\sin^2 t}{4}=1$ ✓

$z=6\sin t\cos t$

$\Rightarrow$ parametrization of the intersection curve; $x=3\cos t$, $y=2\sin t$, $z=6\sin t\cos t$, $t\in[0,2\pi]$

b) The paraboloid $z=4x^2+y^2$ and the parabolic cylinder $y=x^2$

Let $t=x \Rightarrow y=t^2 \Rightarrow z=4t^2+t^4$, $t\in\mathbb{R}$

Q2 The parametric curve C given by

$\rightarrow x=a\cos t\sin t$, $y=a\sin^2 t$, $z=bt$, $t\in\mathbb{R}$

Express the length of C between $t=0$ & $t=T>0$.

$\int_C 1\,ds=$ length of the curve $C=L$

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

$L=\int_0^T\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

$=\int_0^T\sqrt{\underbrace{a^2\cos^2 2t+a^2\sin^2 2t}_{a^2}+b^2}\,dt$

$=\int_0^T\sqrt{a^2+b^2}\,dt=\sqrt{a^2+b^2}\cdot t\Big|_0^T=\sqrt{a^2+b^2}\cdot T$

$\frac{dx}{dt}=a\cdot(-\sin t\sin t+\cos t\cos t)$
$=a(\cos^2 t-\sin^2 t)$
$=a\cos 2t$

$\frac{dy}{dt}=a\,2\sin t\cos t$
$=a\sin 2t$

$\frac{dz}{dt}=b$

0

$$\underbrace{\cos^2 t}_{\frac{x^2}{ay}} + \underbrace{\sin^2 t}_{\frac{y}{a}} = 1$$

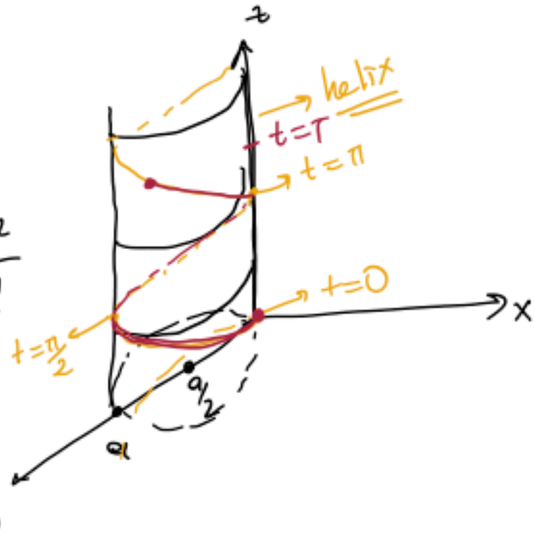
(y)

$$\cos^2 t = \frac{x^2}{a^2 \sin^2 t} = \frac{x^2}{ay}$$

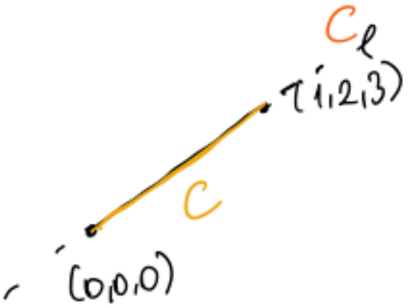
$$x^2 + y^2 = ay$$

$$\underbrace{x^2 + y^2 - ay}\, = 0 \rightarrow x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

$$z = bt$$



Q3) Find $\int_C x e^{yz}\, ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$



$$\ell : (x,y,z) = (0,0,0) + t\vec{v}$$

$$= (0,0,0) + t\,\langle (1,2,3) - (0,0,0) \rangle$$

$$= (t, 2t, 3t) \quad , t \in \mathbb{R}$$

$C: \; x = t,\; y = 2t,\; z = 3t$ where $t \in [0,1]$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$ds = \sqrt{1 + 4 + 9}\, dt$$

$$\int_C x e^{yz}\, ds = \int_0^1 t \cdot e^{2t \cdot 3t} \sqrt{14}\, dt = \frac{\sqrt{14}}{12}\left(e^6 - 1\right)$$

Q4] Evaluate $\int_C \frac{x^2}{y^{4/3}}\, ds$, where $x=t^2$, $y=t^3$ for $1 \le t \le 2$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{4t^2 + 9t^4}\, dt$$

$$\int_C \frac{x^2}{y^{4/3}}\, ds = \int_1^2 \frac{\cancel{t^4}}{\cancel{t^4}} \sqrt{4t^2+9t^4}\, dt = \int_1^2 |t| \sqrt{4+9t^2}\, dt$$

$$= \int_1^2 t\sqrt{4+9t^2}\, dt = \int_{13}^{40} \frac{1}{18} \cdot \sqrt{u}\, du$$

$u = 4+9t^2$

$du = 18t\, dt$

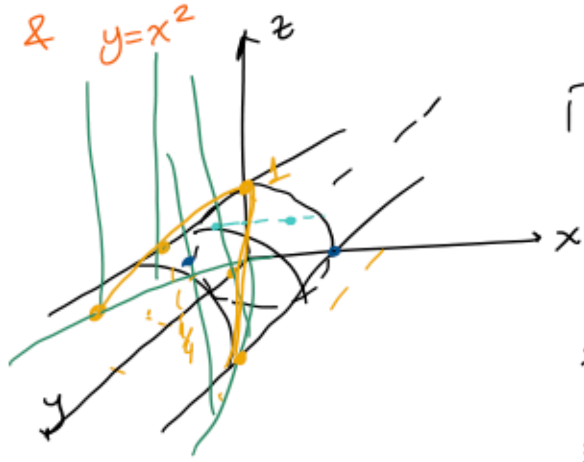
$$= \frac{1}{27}\left(\sqrt{40^3} - \sqrt{13^3}\right) //$$

Q5] Find $\int_C \sqrt{1+4x^2z^2}\, ds$ where C is the curve of intersection of the surfaces $x^2+z^2=1$ & $y=x^2$

$x = \cos t$ $\quad t \in [0, 2\pi]$

$z = \sin t$

$y = \cos^2 t$



$\boxed{z=0}$

$x = 1, -1$

$z = \frac{1}{2}$

$x = \frac{1}{2}, -\frac{1}{2}$

$z = 1$

$x=0, y=0$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{\sin^2 t + 4\sin^2 t\cos^2 t + \cos^2 t}\, dt$$

$$= \sqrt{1 + (2\sin t\cos t)^2}\, dt = \sqrt{1+\sin^2 2t}\, dt$$

2π

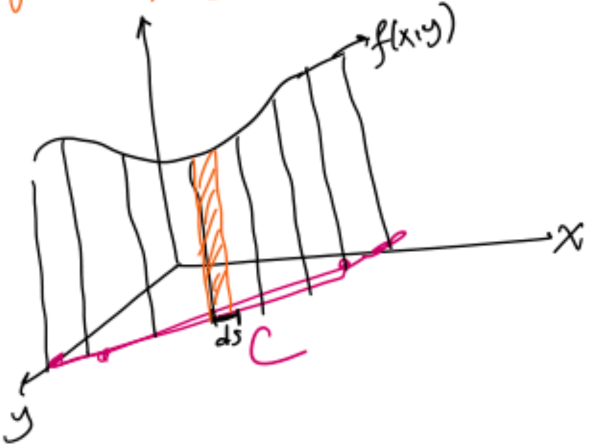
$$\int_C \sqrt{1+4x^2z^2}\, ds = \int_0^{2\pi} \sqrt{1+\underbrace{4\cos^2 t \sin^2 t}_{\sin^2 2t}} \cdot \sqrt{1+\sin^2 2t}\; dt$$

$$= \int_0^{2\pi} |1+\sin^2 2t|\, dt = \int_0^{2\pi} 1+\sin^2 2t\; dt = \int_0^{2\pi} 1+\frac{(1-\cos 4t)}{2}\, dt$$

$$\cos 4t = 1 - 2\sin^2 2t$$

$$= \left(\frac{3}{2}t - \frac{\sin(4t)}{8}\right)\Big|_0^{2\pi} = 3\pi$$

Q6| Find the area of oneside of the wall standing orthogonally on the curve $2x+3y=6$, $0 \le x \le 6$ and beneath the curve on the surface $f(x,y) = 4+3x+2y$



$$\int_C f(x,y) \cdot ds$$

Let $x = x$, $x \in [0,6]$

$$y = \frac{6-2x}{3}$$

$$ds = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1+\frac{4}{9}}\, dx$$

Area of this wall $= \int_C f(x,y)\, ds$

$$= \int_0^6 4+3x+2\left(\frac{6-2x}{3}\right) \frac{\sqrt{13}}{3}\, dx$$

$\Rightarrow$ rest is an exercise

Q7) Show that if the dot product of the velocity and acceleration of a moving particle is positive, then the speed of the particle is increasing

We know that the velocity is a vector, call it $\vec{v}$.

The speed is $|\vec{v}|$

Assume that $\vec{v}.\vec{a} > 0$

Observe that $\vec{v}.\vec{v} = |\vec{v}|^2$

$$\frac{d}{dt}\vec{v}.\vec{v} = \underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}}.\vec{v} + \vec{v}.\underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}} = 2\vec{v}.\vec{a} > 0 \text{ by our assumption}$$

$$\|$$

$$\frac{d}{dt}|\vec{v}|^2 = 2|\vec{v}|.\frac{d|\vec{v}|}{dt}$$

$$\Rightarrow \boxed{2|\vec{v}|.\frac{d|\vec{v}|}{dt} > 0} \text{ since } |\vec{v}| \text{ is always positive}$$

$$\Rightarrow \frac{d|\vec{v}|}{dt} > 0 \Rightarrow |\vec{v}| \text{ is increasing.}$$