Week 12

3 Haziran 2021 Perşembe 08:40

Q1 Let

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Calculate the partial derivatives $f_1(0,0)$, $f_2(0,0)$, $f_{12}(0,0)$, $f_{21}(0,0)$

$$f_1(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2+0^2} - 0}{h} = 1$$

If $(x,y) \neq (0,0)$ then $f_1(x,y) = \dfrac{3x^2(x^2+y^2) - x^3(2x)}{(x^2+y^2)^2} = \dfrac{x^4+3x^2y^2}{(x^2+y^2)^2}$

So $$f_1(x,y) = \begin{cases} \dfrac{x^4+3x^2y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

$$f_{12}(0,0) = \lim_{h \to 0} \frac{f_1(0,0+h) - f_1(0,0)}{h} = \lim_{h \to 0} \frac{0-1}{h} = \lim_{h \to 0} -\frac{1}{h} = \infty$$

$\Rightarrow f_{12}$ dne at $(0,0)$.

$f_2(0,0)$, $f_{21}(0,0)$ $\leftarrow$ left as an exercise.

Q2 Consider the function $f(x,y,z) = x^2y + yz + z^2$

a) Find directional derivative of f at $(1,-1,1)$ in the direction of the vector $\vec{i} + \vec{k}$.

Observe that these are cont $\Rightarrow f(x,y,z)$ is differentiable.

$$\left.\begin{array}{l} f_1(x,y,z) = xy \\ f_2(x,y,z) = x^2+z \\ f_3(x,y,z) = y+2z \end{array}\right\}$$

$$D_u f\Big|_{(1,-1,1)} = \nabla f(1,-1,1)\cdot\frac{\vec{u}}{\|\vec{u}\|} = (-2,2,1)\cdot\frac{(1,0,1)}{\sqrt{2}}$$

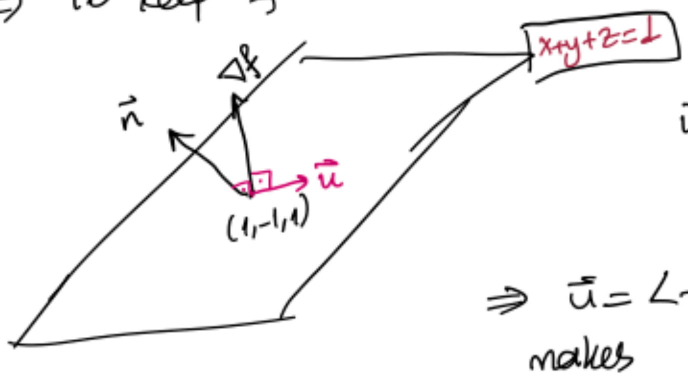
In our case $u = \vec{i}+\vec{k} = (1,0,1)$ $\quad = -\frac{1}{\sqrt{2}}$ //

b) An ant crawling on the plane $x+y+z=1$ through $(1,-1,1)$. Suppose it crawls so as to keep f constant. In what direction is it going as it passes through $(1,-1,1)$.

Let u be any vector, $D_u f = \nabla f\cdot\frac{\vec{u}}{\|\vec{u}\|} = 0$

$$|\nabla f|\cdot\underbrace{\left|\frac{\vec{u}}{\|\vec{u}\|}\right|}_{1}\cos\theta$$

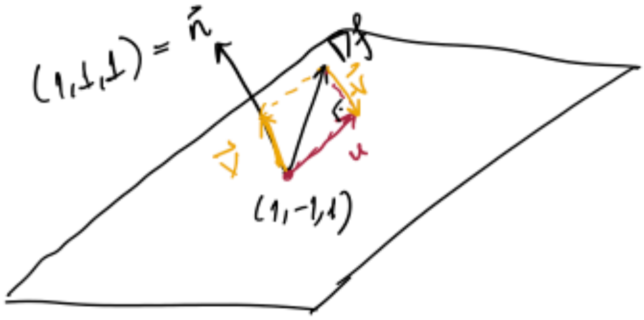
$\Rightarrow$ To keep f as a constant $\theta = \frac{\pi}{2} \Rightarrow$



$$\vec{u} = \underset{(-2,2,1)}{\nabla f}\times\underset{(1,1,1)}{\vec{n}} = \langle -1,-3,4\rangle$$

$\Rightarrow \vec{u} = \langle -1,-3,4\rangle$ is the vector which makes directional derivative 0 so ant can keep f as a constant in this direction.

c) Another ant crawls on the plane $x+y+z=1$, moving in the direction of the greatest rate of increase of f. Find its direction as it goes through $(1,-1,1)$



$x+y+z=1.$

projection of ∇f onto $\vec{n}$:

$$\vec{v} = \frac{\nabla f(1,-1,1)\cdot \vec{n}}{\underbrace{\|\vec{n}\|}_{\sqrt{3}}} \frac{\vec{n}}{\underbrace{\|\vec{n}\|}_{\sqrt{3}}}$$

$$= \frac{(-2,2,1)\cdot(1,1,1)}{3} \cdot \langle 1,1,1\rangle$$

$$\vec{v} = \langle \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3} \rangle$$

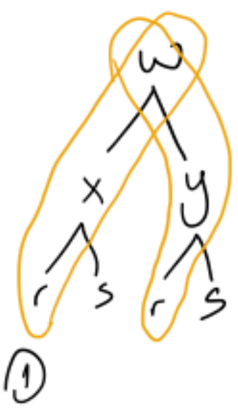
$$\nabla f - \vec{v} = \vec{u}$$

$$(-2,2,1) - (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = \langle -\tfrac{7}{3}, \tfrac{5}{3}, \tfrac{2}{3} \rangle$$ ← the vector on the plane which gives us the greatest rate of change on the plane

Q3) Find $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial s}$ when $r=\pi$ & $s=0$

if $w = \sin(2x-y)$, $x = r+\sin s$, $y = rs$

$w = w(x,y)$, $x = x(r,s)$, $y = (r,s)$ $\Rightarrow$ $w(r,s) = w(x(r,s), y(r,s))$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\left.\frac{\partial w}{\partial r}\right|_{(r,s)=(\pi,0)} = \left(\underbrace{(\cos(2x-y)\cdot 2)\cdot 1}_{2} + \underbrace{\cos(2x-y)\cdot(-1)\cdot s}_{0}\right)\Big|_{(\pi,0)}$$

$$= 2$$

(1) $(r,s) = (\pi,0) \Rightarrow (x,y) = (\pi, 0)$

$\frac{\partial w}{\partial s}$ as an exercise

Q4| Locate & classify the critical pts of the funct...

$$f(x,y) = x^2 - xy + y^2 + 2x + 2y - 4$$

$$\left.\begin{array}{l} f_x(x,y) = 2x - y + 2 \\ f_y(x,y) = -x + 2y + 2 \end{array}\right\} \quad \nabla f = (f_x, f_y) = (0,0)$$

$$\Rightarrow \left.\begin{array}{l} 2x - y + 2 = 0 \\ 2/ -x + 2y + 2 = 0 \end{array}\right] \Rightarrow 3y = -6 \Rightarrow \boxed{y = -2} \Rightarrow \boxed{x = -2}$$

we have only 1 critical point which is $(-2,-2)$.

$$D(x,y) = f_{xx}(-2,-2)\, f_{yy}(-2,-2) - [f_{xy}(-2,-2)]^2$$

$$f_{xx}(x,y) = 2 \qquad f_{xy} = -1$$

$$f_{yy}(x,y) = 2$$

$$D(-2,-2) = 2 \cdot 2 - [-1]^2 = 3$$

Since $D(-2,-2) = 3 > 0$ & $f_{xx}(-2,-2) > 0$ then $f(-2,-2)$ is local minimum value.

Q5| Find the extreme value of $f(x,y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \leq 9$.

$f_x(x,y) = 2x - 3 - y$, $f_y(x,y) = 2y - x$

Since both f_x, f_y are cont, then f is differentiable $\Rightarrow f$ is continuous.

Our region is closed & bounded, then f takes its extreme values on this region.

Observe that there is no singular points.

$$f_x(x,y) = 0 \;\&\; f_y(x,y) = 0 \Rightarrow \left.\begin{array}{l} 2x - y = 3 \\ x = 2y \end{array}\right\} \quad 3y = 3 \Rightarrow \boxed{y = 1}$$

$$\boxed{x = 2}$$

$x^2+y^2 = 4+1 \leq 9 \Rightarrow \boxed{(2,1) \in R}$ where $R: x^2+y^2 \leq 9$

For the boundary, we'll use Lagrange multiplier.

$g(x,y) = k, \quad \nabla f = \lambda \nabla g, \quad \lambda \neq 0$

In our case $g(x,y) = x^2+y^2, \quad k = 9$

$(2x-3-y,\ 2y-x) = \lambda(2x, 2y)$

$\Rightarrow 2x-3-y = \lambda 2x$

$2y-x = \lambda 2y$

If $x=0$, $\left.\begin{array}{l} -3-y=0 \\ 2y = \lambda 2y \end{array}\right\}$ $y=-3, \lambda=1 \Rightarrow \boxed{(0,-3)}$

If $y=0$, $2x-3 = \lambda 2x \Rightarrow -3=0$ ↯ No solution.

$-x=0$

If $x \neq 0$ & $y \neq 0$;

$\lambda = \frac{2x-3-y}{2x} = \frac{2y-x}{2y} \Rightarrow \cancel{1} - \frac{3}{2x} - \frac{y}{2x} = \cancel{1} - \frac{x}{2y}$

$\frac{x}{2y} - \frac{3}{2x} - \frac{y}{2x} = 0$

$(x) \quad (y) \quad (y)$

$\frac{x^2-3y-y^2}{2xy} = 0$

$\Leftrightarrow$

$x^2 - 3y - y^2 = 0$

Now, we'll just check $x^2 - 3y - y^2 = 0 \cap x^2 + y^2 = 9$

$x^2 = 9 - y^2$

$\Rightarrow 9 - 2y^2 - 3y = 0 \Rightarrow (2y-3)(y+3) = 0$

$\Rightarrow y = \frac{3}{2}, \underbrace{y = -3}$

we already found this one in the case $x=0$.

$x = \mp\sqrt{9-y^2} \Rightarrow x = \mp\sqrt{\frac{27}{4}}$

$= \mp \frac{3\sqrt{3}}{2}$

$\Rightarrow \boxed{\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right), \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)}$

We have 4 candidates; $(2,1)$, $(0,-3)$, $\left(\mp\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

$f(2,1) = 4+1-6-2 = \boxed{-3} \longrightarrow f(2,1)$ is extreme minimum.

$f(0,-3) = \boxed{9}$

$f\left(\mp\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

$f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = \frac{27}{4} + \frac{9}{4} + \frac{9\sqrt{3}}{\underset{(2)}{2}} + \frac{9\sqrt{3}}{4} \rightarrow f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ is extreme maximum.

$= \frac{36 + 27\sqrt{3}}{4} = \boxed{9 + \frac{27\sqrt{3}}{4}}$

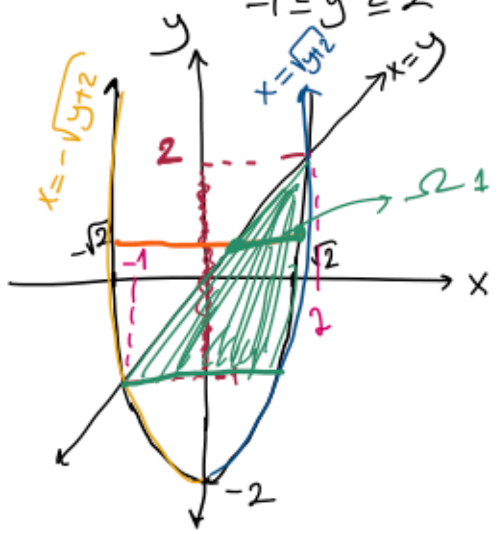
$f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = \frac{27}{4} + \frac{9}{4} - \frac{9\sqrt{3}}{2} - \frac{9\sqrt{3}}{4}$

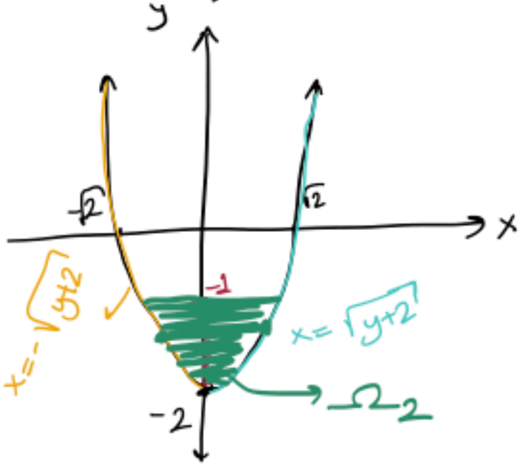
$= \boxed{9 - \frac{27\sqrt{3}}{4}}$

Q6) Evaluate $\underbrace{\displaystyle\int_{-1}^{2}\int_{y}^{\sqrt{y+2}} e^{2x^3-3x^2-12x}\,dx\,dy}_{I_1} + \underbrace{\displaystyle\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}} e^{2x^3-3x^2-12x}\,dx\,dy}_{I_2}$

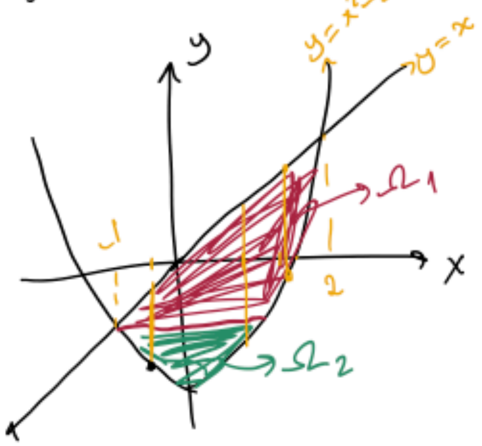
For I_1: $y \le x \le \sqrt{y+2}$, $-1 \le y \le 2$

I_2: $-\sqrt{y+2} \le x \le \sqrt{y+2}$, $-2 \le y \le -1$





If I take the union of these two regions Ω_1, Ω_2, $\Omega_1 \cup \Omega_2$ is connected.



$$\iint_{\Omega_1 \cup \Omega_2} e^{2x^3-3x^2-12x}\,dA$$

$$\parallel$$

$$\int_{-1}^{2}\int_{x^2-2}^{x} e^{2x^3-3x^2-12x}\,dy\,dx$$

$$\parallel$$

$$\int_{-1}^{2} e^{2x^3-3x^2-12x}\cdot y\Big|_{x^2-2}^{x}\,dx$$

2 $\parallel$

$$2x^3-3x^2-12x=u \leftarrow \int_{-1} e^{2x^3-3x^2-12x}\,\underbrace{(x-x^2+2)\,dx}_{-\frac{1}{6}du}$$

$$6x^2-6x-12\,dx=du$$

$$x^2-x-2\,dx=-\frac{1}{6}du$$

$$x=-1,\ -2-3+12=7=u$$

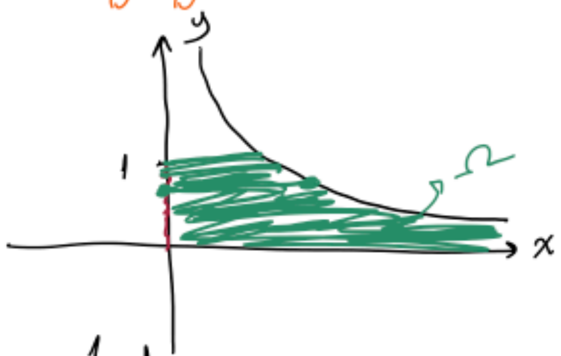
$$x=2,\ 16-12-24=-20$$

$$\int_{7}^{-20} -e^u\cdot\frac{1}{6}\cdot du = \int_{-20}^{7}\frac{e^u}{6}\,du$$

$$=\frac{1}{6}\cdot\left(e^7-\frac{1}{e^{20}}\right)$$

Q7| Sketch the region of integration and evaluate the double integral

a) $\displaystyle\int_0^1\int_0^{1/y} y\,e^{xy}\,dx\,dy \Rightarrow$ $0\le x\le\frac{1}{y}$, $0\le y\le 1$

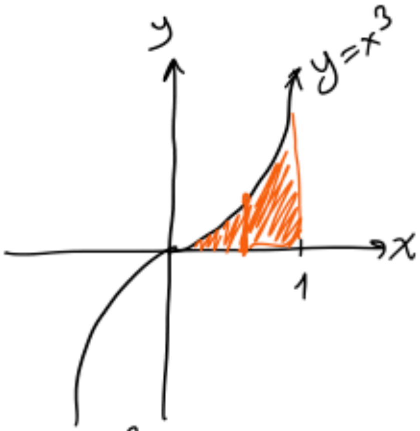


Observe that we cannot change the order of iteration, which means that we can evaluate this integral in this order only.

$$\int_0^1\int_0^{1/y} y\,e^{xy}\,dx\,dy = \int_0^1 y\cdot\left(\frac{e^{xy}}{y}\right)\Bigg|_{x=0}^{x=\frac{1}{y}} dy$$

$$=\int_0^1 e-1\,dy = y(e-1)\Big|_0^1 = e-1$$

b) $\displaystyle\int_0^1\int_0^{x^3} e^{\frac{y}{x}}\,dy\,dx \Rightarrow$ $0\le y\le x^3$, $0\le x\le 1$



Since dx dy order will give us integration of $e^{\frac{y}{x}}$, this integral can be evaluated in this order only eventhough our region is bounded.

$$\int_0^1 \int_0^{x^3} e^{y/x}\, dy\, dx = \int_0^1 \left(\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right) \Bigg|_{y=0}^{y=x^3} dx$$

$$= \int_0^1 x e^{x^2} - x \quad dx = \frac{e}{2} - 1 \;//$$