REC - VIII

Recall:

vertical asymptote: $y=f(x)$ has vertical asymptote at $x=a$ if $\lim_{x\to a^-} f(x)=\mp\infty$ or $\lim_{x\to a^+} f(x)=\mp\infty$

or both.

Horizontal asymptote: $y=f(x)$ has a horizontal asymptote at $y=L$ if either $\lim_{x\to\infty} f(x)=L$

or $\lim_{x\to-\infty} f(x)=L$ or both.

Oblique asymptote: The straigth line $y=ax+b$ $(a\neq 0)$ is an oblique asymptote if either $\lim_{x\to\infty}(f(x)-(ax+b))=0$

or $\lim_{x\to-\infty}(f(x)-(ax+b))=0$ or both.

Sketching the Graph of a Function

1) Find the domain of $f(x)$
2) Find x-intercept & y-intercept
3) Find its asymptotes
4) Find & classify its critical points & singular pts.
5) Find the intervals where $f(x)$ is increasing & decreasing.

6) Determine the inflection points

7) Find the intervals where $f(x)$ is concave up or concave down.

#1: Sketch the graph of the following functions:

(a) $f(x) = 2x^{5/3} - 5x^{4/3}$

① $Dom f = \mathbb{R}$

② y-intercept: For $x=0$, $y=0$. $(0,0)$

x-intercept: $2x^{5/3} - 5x^{4/3} = 0$.

$\Leftrightarrow \quad x^{4/3}(2x^{1/3} - 5) = 0 \Leftrightarrow x=0$ & $x^{1/3} = 5/2$

$(0,0)$ $\Downarrow$ already found!

$\Downarrow$ $x = \frac{125}{8}$ $\left(\frac{125}{8}, 0\right)$

③ vertical asymptote:

Since $\forall a \in Dom f = \mathbb{R}$, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$= \lim_{x \to a} f(x) = L \in \mathbb{R}, (\neq \mp\infty)$, f has no vertical asymptote.

horizontal asymptote: $\lim_{x \to \mp\infty} f(x) = \lim_{x \to \mp\infty} 2x^{5/3} - 5x^{4/3}$

$= \mp\infty \ (\neq L \in \mathbb{R})$ Therefore, f has no horizontal asymptote.

<u>oblique asymptote:</u>

$$\lim_{x\to\mp\infty} \left(f(x) - (ax+b)\right) = \lim_{x\to\mp\infty} 2x^{5/3} - 5x^{4/3} - ax - b.$$

$= \mp\infty \ (\neq 0)$ Therefore, f has <u>no</u> oblique asymptote.

(4) $f'(x) = \frac{10}{3} x^{2/3} - \frac{20}{3} x^{1/3} = \frac{10}{3} x^{1/3}\left(x^{1/3} - 2\right)$

Observe that f has no singular points.

$f'(x) = 0 \iff x = 0 \ \& \ x^{1/3} = 2 \iff x = 8.$

critical pts of f.

(5).

		0		8	
f'	$+$		$-$		$+$
f	$\nearrow$ increasing		$\searrow$ decreasing		$\nearrow$ increasing.

(6) $f''(x) = \frac{20}{9} x^{-1/3} - \frac{20}{9} x^{-2/3} = \frac{20}{9} x^{-1/3}\left(1 - x^{-1/3}\right).$

Observe that f'' is not defined at the point

<u>$x=0$ (double)</u>.

$f''(x) = 0 \iff 1 - x^{-1/3} = 0 \iff x^{-1/3} = 1.$

$\iff x = 1$ is a candidate for an inflection.

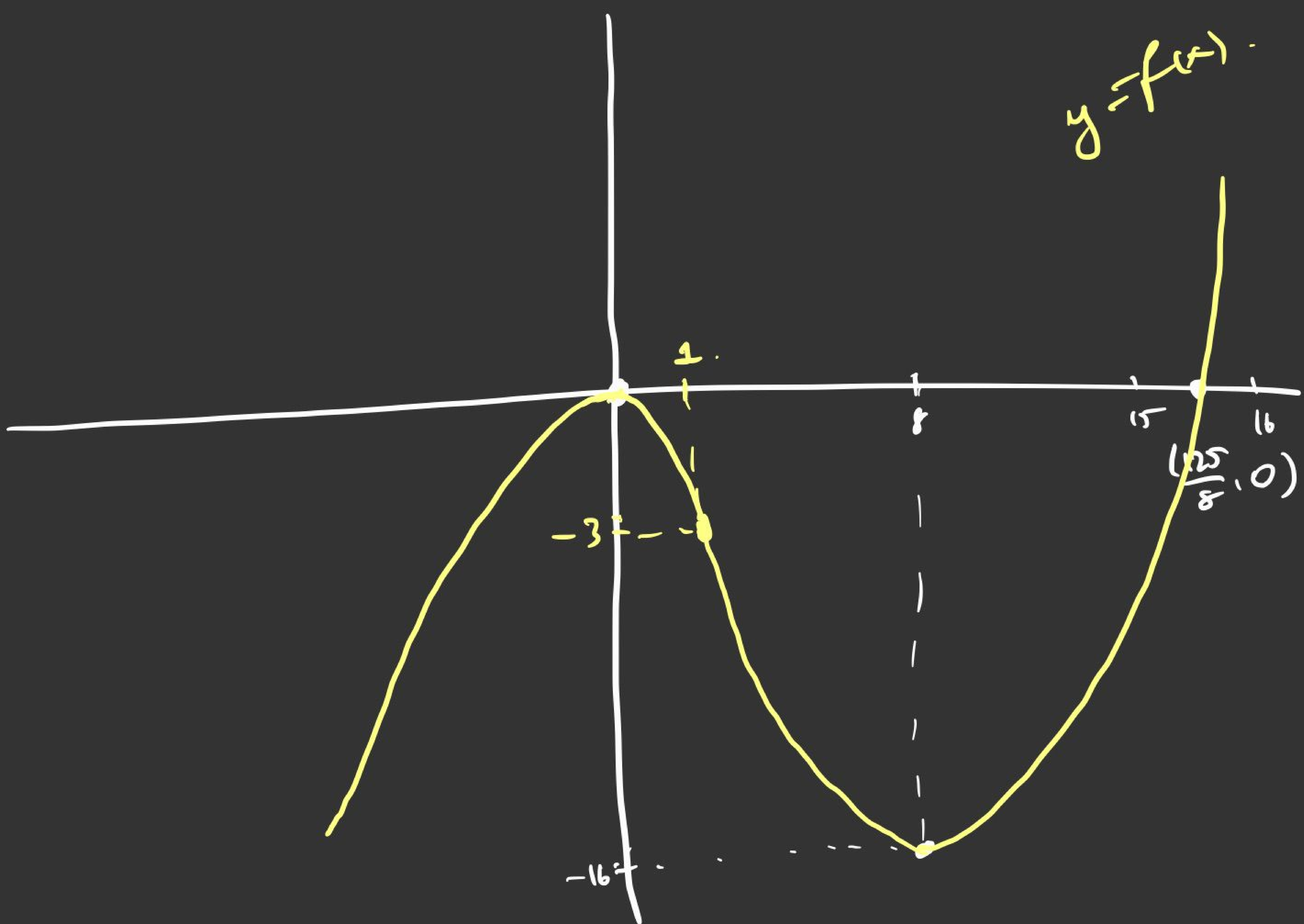
Since at $x=1$ f has a tangent line (since $f'(1)$ exists). & $f''(1) = 0$, $x=1$ is an inflection point.

(7)

		0		1	
f''	$-$		$-$		$+$
f	$\cap$		$\cap$		$\cup$
	concave down		conc. down		concave up.

$f(8) = -16.$

$f(1) = -3.$



(b) $f(x) = \frac{x^2-2x+2}{x-1} = \frac{(x-1)^2+1}{x-1}$.

① $Dom f = \mathbb{R} \setminus \{1\}$.

② y-intercept: for $x=0$, $y=-2$ $(0,-2)$.

x-intercept: $\frac{x^2-2x+2}{x-1} = 0 \iff x^2-2x+2=0$

$\iff (x-1)^2+1=0$. no such $x \in Dom f$.

So f has no x-intercept.

③ vertical: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{(x-1)^2+1}{x-1} = -\infty$.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-1)^2+1}{x-1} = \infty$.

So, at $x=1$ f has a vertical asymptote.

horizontal: $\lim_{x\to \mp\infty} f(x) = \lim_{x\to\mp\infty} \frac{x^2-2x+2}{x-1} \underset{L'H}{\overset{(\infty/\infty)}{=}}$

$= \lim_{x\to\mp\infty} \frac{2x-2}{1} = \mp\infty \; (\neq L)$

So, f has no horizontal asymptote.

oblique: $\lim\limits_{x \to \mp\infty} \left(f(x) - (ax+b)\right) \overset{?}{=} 0$

$$\lim_{x \to \mp\infty} \left(\frac{(x-1)^2 + 1}{x-1} - (ax+b) \right)$$

$$= \lim_{x \to \mp\infty} \left((x-1) + \frac{1}{x-1} - (ax+b) \right) = 0.$$

$\iff \quad ax+b = x-1.$

Therefore, at $y = x-1$, f has an oblique asymptote.

④ $f'(x) = \dfrac{(2x-2)(x-1) - (x^2-2x+2)\cdot 1}{(x-1)^2} = \dfrac{\overset{x^2-2x}{\overline{x(x-2)}}}{(x-1)^2}.$

Observe that at $x=1$, f has a singular pt. (double)

$f'(x) = 0 \iff x=0$ & $x=2$. critical pts of f.

⑤

x		0		1		2	
f'	$+$		$-$		$-$		$+$
f	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

$f(2) = 2.$

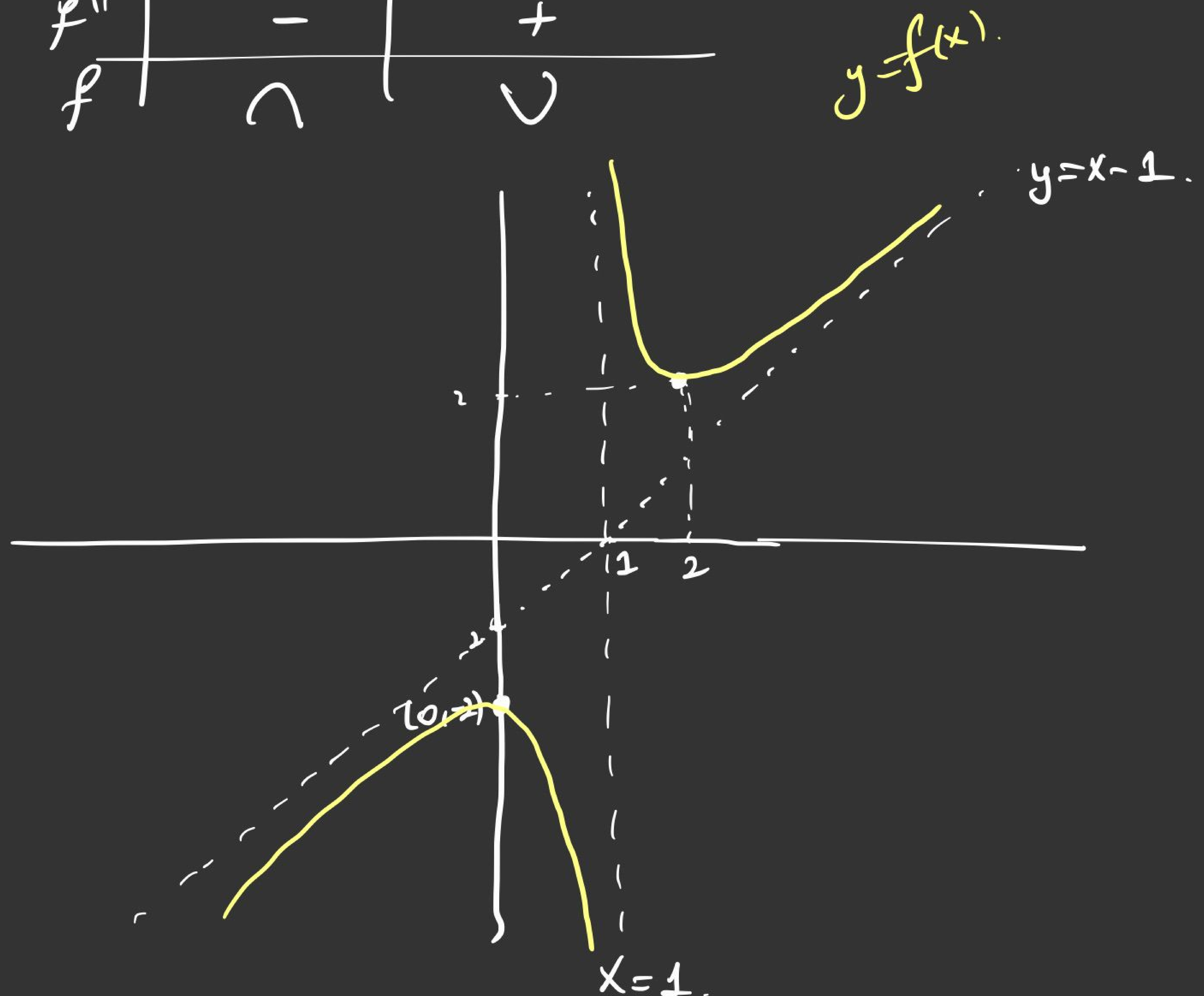
⑥ $f''(x) = \dfrac{(2x-2)(x-1)^2 - (x^2-2x)\cdot 2(x-1)}{(x-1)^4} = \dfrac{2}{(x-1)^3}$.

Observe that at $\underline{\underline{x=1}}$ (single), f'' is not defined.

But f has no inflection point.

⑦

		1	
f''	$-$		$+$
f	$\cap$		$\cup$



(c) $f(x) = \dfrac{x}{\ln x}$

① $\text{Dom} f = (0, \infty) \setminus \{1\}$

② Observe that f has no x-intercept since

$0 = \dfrac{x}{\ln x}$, no such x.

Also no y-intercept since $0 \notin \text{Dom} f$.

③ vertical asymptote:

$\lim_{x \to 1^-} \dfrac{x}{\ln x} = -\infty$, $\lim_{x \to 1^+} \dfrac{x}{\ln x} = \infty$.

So, at $x=1$ f has vertical asymptote.

horizontal $\lim_{x \to \mp\infty} \dfrac{x}{\ln x} \underset{L'H}{\overset{(\frac{\infty}{\infty})}{=}} \lim_{x \to \mp\infty} \dfrac{1}{\frac{1}{x}} = \mp\infty$. $(\neq L)$.

$= x$.

So, f has no horizontal asymptote.

oblique: $\lim_{x \to \mp\infty} \left(f(x) - (ax+b) \right)$

$= \lim_{x \to \mp\infty} \left(\dfrac{x}{\ln x} - (ax+b) \right) = \lim_{x \to \mp\infty} \dfrac{x - ax \ln x - b \ln x}{\ln x}$

(∞/∞)

$\underset{L'H}{=} \lim_{x \to \mp\infty} \dfrac{1 - a\ln x - a - \frac{b}{x}}{\frac{1}{x}} = \lim_{x \to \mp\infty} \underbrace{x}_{\mp\infty} \big(1 - \underbrace{a \ln x}_{\mp\infty} - a - \underbrace{\tfrac{b}{x}}_{0} \big) = \mp\infty$

$\neq 0$.

So f has no oblique asymptote.

(4) $$f'(x) = \frac{\ln x - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}.$$

Observe $x=1$ (double) is a singular pt.

Also at $x=e$, f has a critical point.

(5)

		1		e	
f'	$-$		$-$		$+$
f	$\searrow$		$\searrow$		$\nearrow$

(6) $$f''(x) = \frac{\frac{1}{x}\cdot(\ln x)^2 - (\ln x - 1)\cdot 2\ln x \cdot \frac{1}{x}}{(\ln x)^4} \to \frac{2-\ln x}{(\ln x)^3}$$

Observe that at $x=1$ (single), f'' is not defined.

$f''(e^2)=0$, & since f has a tangent line at $x=e^2$ ($f'(e^2)$ exists), therefore $x=e^2$ is an inflection pt.

7

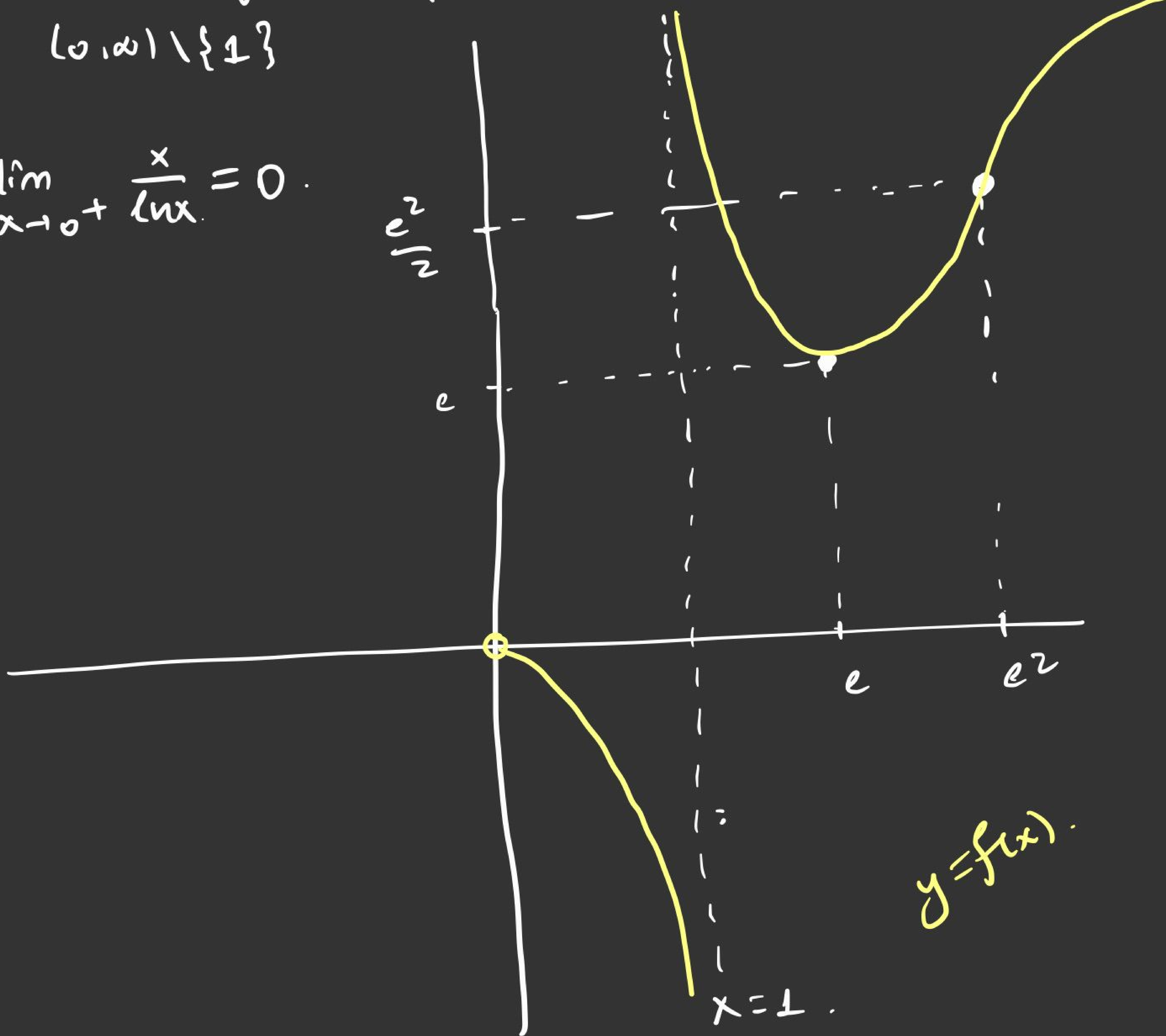
		1		e^2	
f''	$-$		$+$		$-$
f	$\cap$		$\cup$		$\cap$

$(0,\infty)\setminus\{1\}$

$$\lim_{x\to 0^+}\frac{x}{\ln x}=0.$$

$f(e)=e.$

$$f(e^2)=\frac{e^2}{\ln e^2}=\frac{e^2}{2}$$



(d) $f(x) = \dfrac{3x^2+1}{x^2-1}$

① $\text{Dom} f = \mathbb{R} \setminus \{1, -1\}$.

② y-intercept: For $x=0$, $y=-1$. $(0,-1)$.

Observe that f has no x-intercept since

$3x+1 \neq 0$

③ vertical asymptote:

$\lim\limits_{x\to 1^+} \dfrac{3x^2+1}{x^2-1} = +\infty$, $\lim\limits_{x\to 1^-} \dfrac{3x^2+1}{x^2-1} = -\infty$.

$\lim\limits_{x\to (-1)^+} \dfrac{3x^2+1}{x^2-1} = -\infty$. $\lim\limits_{x\to (-1)^-} \dfrac{3x^2+1}{x^2-1} = \infty$.

So, at $x=1$ & $x=-1$ f has a vertical asymptote.

horizontal: $\lim\limits_{x\to \mp\infty} \dfrac{3x^2+1}{x^2-1} \underset{L'H}{\overset{(\infty/\infty)}{=}} \lim\limits_{x\to\mp\infty} \dfrac{6\cancel{x}}{2\cancel{x}} = \underline{\underline{3}}$.

At $y=3$, f has a horizontal asymptote.

oblique asymptote:

$\lim\limits_{x\to\mp\infty} \left(\dfrac{3x^2+1}{x^2-1} - (ax+b) \right) = \lim\limits_{x\to\mp\infty} \dfrac{3x^2+1-ax^3-\ldots}{x^2-1}$.

$\left(\frac{\infty}{\infty}\right)$

$\underset{L'H}{=} \ \cdots\cdots = \mp\infty \quad (\neq 0)$

So no oblique asymptote.

④ $$f'(x) = \frac{6x.(x^2-1) - (3x^2+1).2x}{(x^2-1)^2}$$

$$= \frac{-8x}{(x-1)^2(x+1)^2}$$

So $\underline{x=1}$ & $x=-1$ are singular points. (both are $\underline{\text{double}}$)

At $x=0$, f has a critical pt.

⑤

	-1		0		1	
f'	$+$		$+$		$-$	$-$
f	$\nearrow$		$\nearrow$		$\searrow$	$\searrow$

⑥ $$f''(x) = \frac{-8.(x^2-1)^2 - (-8x).2(x^2-1).2x}{(x-1)^4(x+1)^4}$$

$$= \frac{8(3x^2+1)}{(x-1)^3(x+1)^3}$$

Observe that at $x=1$ & $x=-1$ f'' is not defined.

(both are single)

Also, $f''(x) \neq 0$ $\forall x \in \text{Dom} f$.

So f has no inflection points.

(7)

		-1		1	
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$