REC - VII

#1: Find all local and absolute extrema points of the following functions:

(a) $f(x) = x^4 + 2x^3 - 2x^2 - 6x + 1$.

$f'(x_0) = 0$.

$\Uparrow$

(*) A function f has local extrema values at critical points, singular points and endpoints (of a closed interval) $[a, b]$

$\Downarrow$

$f'(x)$ is not defined at x_0.

$$f'(x) = 4x^3 + 6x^2 - 4x - 6$$

Observe that since f' is defined everywhere, f has no singular points. Also, in this question we don't have endpoints too.

$$f'(x) = 0 \iff 4x^3 + 6x^2 - 4x - 6 = 0.$$

$$\iff 2x^2(2x+3) - 2(2x+3) = 0$$

$$\iff 2.(x^2-1)(2x+3) = 0 \iff 2(x-1)(x+1)(2x+3) = 0.$$

$$\iff x = 1, \quad x = -1, \quad x = -3/2. \rightarrow \text{critical points of } f.$$

$\rightarrow f(1) = 1 + \cancel{2} - \cancel{2} - 6 + 1 = -4$

$f(-1) = \cancel{1} - \cancel{2} - 2 + 6 + \cancel{1} = 4.$

$\rightarrow f\left(-\frac{3}{2}\right) = -\frac{317}{16}$

		$-3/2$		-1		1	
$\rightarrow f'$	$-$		$+$		$-$		$+$
f	$\searrow$	$\cup$	$\nearrow$	$\cap$	$\searrow$	$\cup$	$\nearrow$
		local min.		local max		local min.	

Recall

Thm: Existence of Extreme Values on Open Intervals:

If f is cont. on the open interval (a,b) and if

$\lim_{x\to a^+} f(x) = L$ & $\lim_{x\to b^-} f(x) = M$ then the followings hold:

(i) If $f(u) > L$ & $f(u) > M$ for some $u \in (a,b)$

then f has an absolute max on (a,b).

(ii) If $f(v) < L$ & $f(v) < M$ for some $v \in (a,b)$

then f has an absolute min. on (a,b).

$\mathbb{R} = (-\infty, \infty)$.

Observe that f is cont.

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^4 + 2x^3 - 2x^2 - 6x + 1) = \infty$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^4 + 2x^3 - 2x^2 - 6x + 1) = \infty$.

Observe that this function cannot have absolute max. on $\mathbb{R}$. Also observe that there is some $r \in (-\infty, \infty)$ (you can pick any $r \in \mathbb{R}$).

$f(r) < \infty = \lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x)$, therefore f has absolute minimum on $\mathbb{R}$. Since, here, f realizes its abs. min value at its critical point. f has abs min. value $-\frac{317}{16}$ at the point $x = -\frac{3}{2}$.

(b) $f(x) = \arctan(x - \sqrt{x})$ on $[0, 5]$.

[Thm: If f is a continuous function on a closed interval then it has abs. max & abs. min. values.]

Observe that f is cont. By thm, f has abs. max & abs. min.

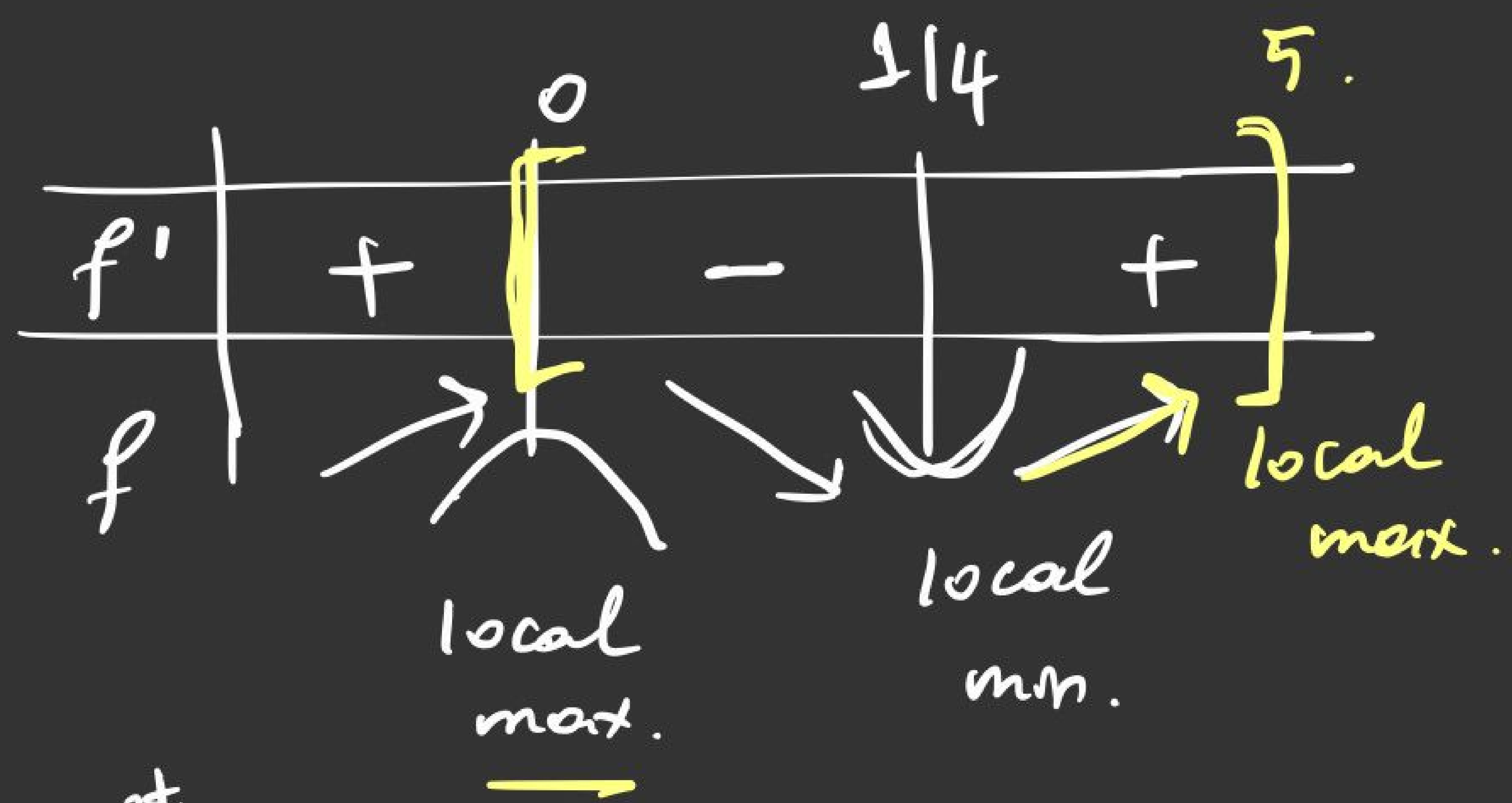
$$\underline{f'(x)} = \frac{1}{1+(x-\sqrt{x})^2} \cdot (x-\sqrt{x})' = \frac{1}{1+(x-\sqrt{x})^2}\left(1 - \frac{1}{2\sqrt{x}}\right).$$

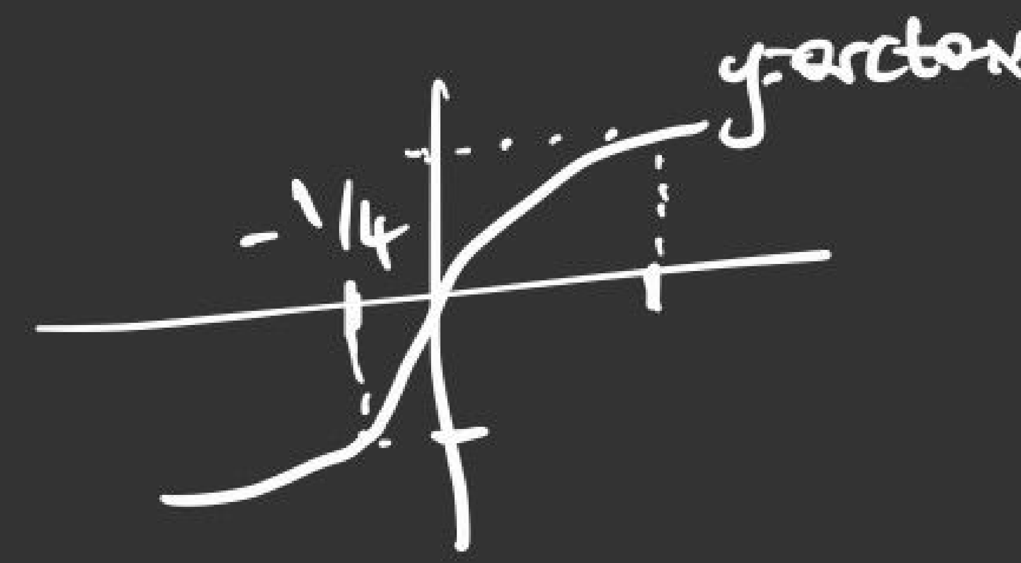
We see that $x = 0$ is a singular point of f.

$$f'(x) = 0 \iff \underbrace{\frac{1}{\underbrace{1+(x-\sqrt{x})^2}_{>0}}}_{\neq 0} \cdot \left(1 - \frac{1}{2\sqrt{x}}\right) = 0$$

$$\Longleftrightarrow \quad 1 - \frac{1}{2\sqrt{x}} = 0 \Longleftrightarrow \sqrt{x} = \frac{1}{2} \Longleftrightarrow x = \frac{1}{4}$$

$\downarrow$ critical pt of f.





sing. pt

$f(0) = \arctan 0 = 0.$

cr. pt. $f\left(\frac{1}{4}\right) = \arctan\left(\frac{1}{4} - \frac{1}{2}\right) = \arctan\left(-\frac{1}{4}\right) < 0.$

end pt. $f(0) = 0$

$f(5) = \arctan(5 - \sqrt{5}) > 0.$

At $x = \frac{1}{4}$ f has abs min value $\arctan\left(-\frac{1}{4}\right)$

At $x = 5$ f has abs max value $\arctan(5 - \sqrt{5})$.

#2: Find the intervals of concavity of $f(x)=x^2 \cdot e^{-x^2}$.

and locate any inflection points.

inflection point: x_0 is an inflection point if

(i) $y=f(x)$ has a tangent line at $x=x_0$. $\Leftrightarrow$ $f'(x_0)$ exists OR $\pm\infty$.

(ii) At $x=x_0$, the concativity of f changes $\Leftrightarrow f''(x_0)=0$.

Thm: (i) If $f''(x)>0$ on the interval I then f is concave up on I.

(ii) If $f''(x)<0$ on I, then f is concave down on I.

$$f'(x) = 2x \cdot e^{-x^2} + x^2 \cdot e^{-x^2} \cdot (-2x)$$

$$= e^{-x^2}(2x - 2x^3).$$

$$f''(x) = e^{-x^2} \cdot (-2x) \cdot (2x-2x^3) + e^{-x^2} \cdot (2-6x^2).$$

$$= e^{-x^2} \cdot (4x^4 - 10x^2 + 2).$$

$$f''(x) = 0 \Leftrightarrow \underbrace{e^{-x^2}}_{\neq 0} \cdot (4x^4 - 10x^2 + 2) = 0.$$

$$\Leftrightarrow 4x^4 - 10x^2 + 2 = 0 \underset{x^2=t}{\Leftrightarrow} 4t^2 - 10t + 2 = 0.$$

$\Leftrightarrow \quad \underline{\underline{t_{1,2}}} = \dfrac{10 \mp \sqrt{68}}{8} = \dfrac{10 \mp 2\sqrt{17}}{8} = \underline{\underline{\dfrac{5 \mp \sqrt{17}}{4}}}.$

$\Delta = 100 - 32$
$= 68.$

$t_{1,2} = \dfrac{-b \mp \sqrt{\Delta}}{2a}$

$\Leftrightarrow \quad x_1 = \dfrac{\sqrt{5+\sqrt{17}}}{2}, \quad x_2 = \dfrac{\sqrt{5-\sqrt{17}}}{2}, \quad x_3 = -\dfrac{\sqrt{5+\sqrt{17}}}{2},$

$x = \mp\sqrt{t}$

$x_4 = -\dfrac{\sqrt{5-\sqrt{17}}}{2}.$

$f''(x_i) = 0$ $\quad i = 1, \dots, 4$. Also since $f'(x_i)$ $i = 1, \dots, 4$ exists, x_i's $(i = 1, \dots, 4)$ are inflection points.

	$-\infty$	$-\frac{\sqrt{5+\sqrt{17}}}{2}$	$-\frac{\sqrt{5-\sqrt{17}}}{2}$	$\frac{\sqrt{5-\sqrt{17}}}{2}$	$\frac{\sqrt{5+\sqrt{17}}}{2}$	∞
f''	$+$	$-$	$+$	$-$	$+$	
f	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	
	conc. up.	conc. down	conc up	conc. down	conc up.	

#3: Find all asymptotes of $f(x)=\frac{\sqrt{2x+1}}{3x+1}$.

Recall:

vertical asymptotes: $y=f(x)$ has a vertical asymptote at $x=a$ if $\lim_{x\to a^-} f(x)=\mp\infty$ or $\lim_{x\to a^+} f(x)=\mp\infty$ or both.

horizontal asymptote: $y=f(x)$ has a horizontal asymptote at $y=L$ if either $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$ or both.

oblique asymptote: The straight line $y=ax+b$ $(a\neq 0)$ is an oblique asymptote of $y=f(x)$ if either

$\lim_{x\to\infty} (f(x)-(ax+b))=0$ or $\lim_{x\to-\infty} (f(x)-(ax+b))=0$ or both.

Observe that at the point $x=-\frac{1}{3}$

$$\lim_{x\to-\frac{1}{3}} f(x)=\lim_{x\to-\frac{1}{3}} \frac{\sqrt{2x+1}}{3x+1}=\infty.$$

Therefore $x=-\frac{1}{3}$ is a vertical asymptote.

Also, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \underbrace{\frac{\sqrt{2x+1}}{3x+1}}_{} \overset{(\infty/\infty)}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{2x+1}}\cdot\cancel{2}}{3}$

$= \lim_{x\to\infty} \frac{1}{3\sqrt{2x+1}} = \underline{0}.$

Therefore $y=0$ is a horizontal asymptote of f.

$\lim_{x\to\infty} (f(x) - (ax+b)) = \lim_{x\to\infty} \left(\frac{\sqrt{2x+1}}{3x+1} - (ax+b)\right)$

$= \underbrace{\lim_{x\to\infty} \frac{\sqrt{2x+1}}{3x+1}}_{0} - \underbrace{\lim_{x\to\infty} ax+b}_{\infty} = -\infty. \ (\neq 0)$

Therefore f does not have an oblique asymptote.

#4: Let f be a continuous function and its derivatives exist everywhere. Also let $f(-1)=0$, $f(0)=2$, $f(1)=1$

$f(2)=0$, $f(3)=1$ and $\lim_{x\to\mp\infty} \left(f(x) + \frac{1-x}{-(x-1)}\right) = 0.$

Assume that $f'(x) > 0$ on $(-\infty,-1)$, $(-1,0)$ & $(2,\infty)$

and $f'(x) < 0$ on $(0,2)$ such that $\lim_{x\to-1} f'(x) = \infty$.

Suppose also that $f''(x) > 0$ on $(-\infty,-1)$ & $(1,3)$

and $f''(x) < 0$ on $(-1, 1)$ & $(3, \infty)$. Sketch the graph of f and identify any critical points.

Observations: ① $y = x - 1$ is an oblique asymptote.

② $x = -1$ is a singular point.

③ For $x < 0$ $f'(x) > 0$
For $x > 0$ $f'(x) < 0$ $\xrightarrow{f' \text{ is cont.}}$ For $x = 0$ we have $f'(x) = 0$.
since f'' exists

Therefore, $x = 0$ is a critical pt.

④ For $x > 2$ $f'(x) > 0$
For $x < 2$ $f'(x) < 0$ $\xrightarrow{f' \text{ is cont.}}$ For $x = 2$, $f'(x) = 0$.

Therefore, $x = 2$ is also critical pt.

⑤.

		-1		0		2	
f'	+		+		−		+
f	↗		↗	∩ local max.	↘	∪ local min.	↗

(6) For $x < -1$ $f''(x) > 0$ $\xLeftrightarrow{f'' \text{ is cont.}}$ For $x = -1$

For $x > -1$ $f''(x) < 0$. f''' exists. $f''(x) = 0$.

Also since $\lim_{x \to -1} f'(x) = \infty$, at $x = -1$ f has a vertical tangent line, $x = -1$ is an inflection point.

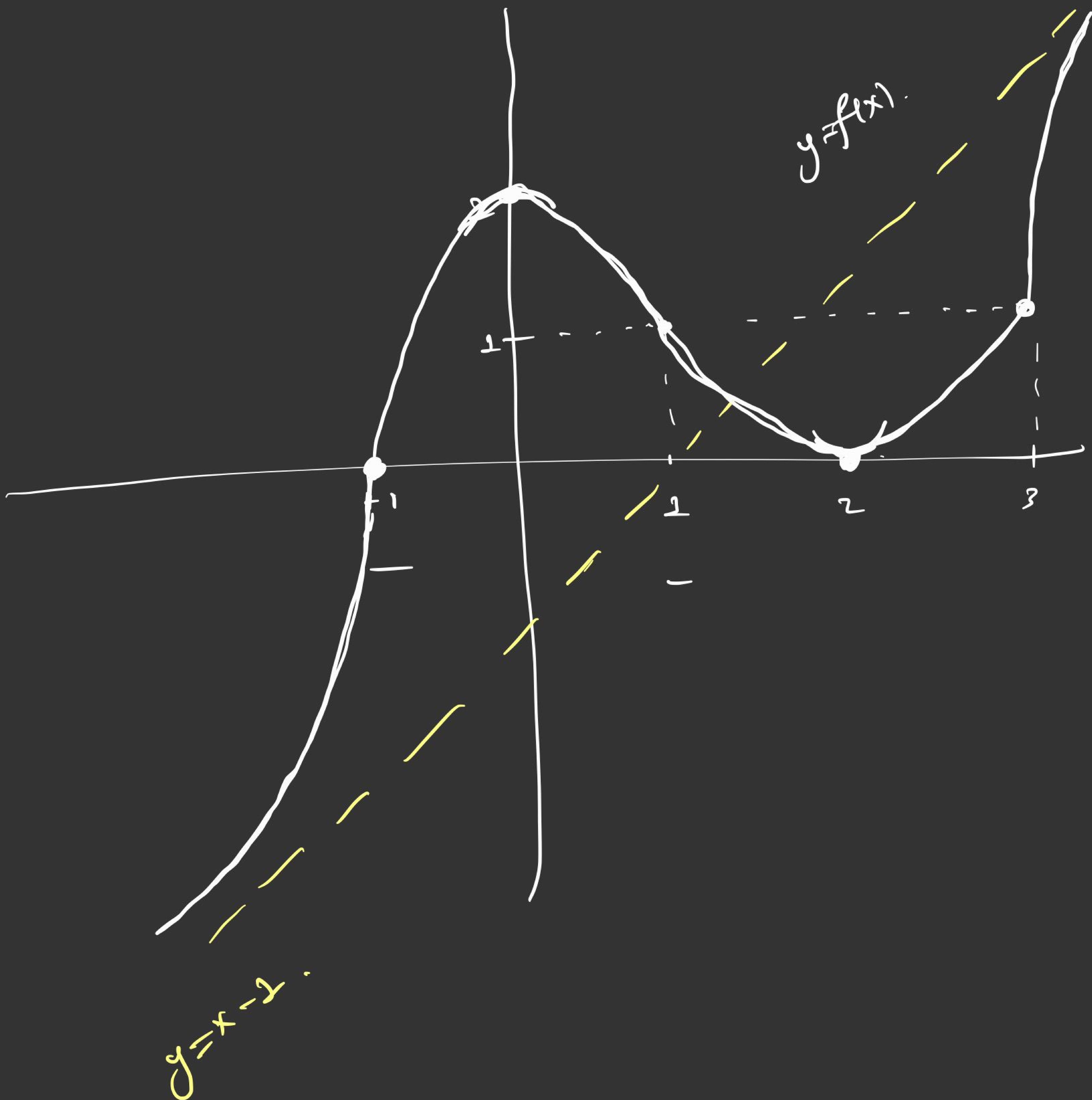
(7) For $x < 1$ $f''(x) < 0$ $\xLeftrightarrow{f'' \text{ is cont}}$ $f''(1) = 0$.

For $x > 1$ $f''(x) > 0$ since f''' exists.

And also $f'(1)$ exists, $x = 1$ is also an inflection pt.

(8) Also observe that $x = 3$ is an inflection pt.

(Exercise)



y=f(x).
1
-1
1
2
3
y=x-1.