REC-VII

#1: Find all local and absolute extrema points of the following functions:

(a) $f(x) = x^4 + 2x^3 - 2x^2 - 6x + 1$.

⊛ A function f has local extreme values at critical points, singular points and endpoints (of a closed interval)

critical points $\Rightarrow f'(x_0) = 0$.

singular points $\Downarrow$ where $f'(x_0)$ is undefined.

endpoints $[a,b]$.

$$f'(x) = 4x^3 + 6x^2 - 4x - 6.$$

$$f'(x) = 0 \iff 4x^3 + 6x^2 - 4x - 6 = 0 \iff$$

$$2x^2(2x+3) - 2(2x+3) = 0 \iff (2x^2 - 2)(2x+3) = 0.$$

$$\iff 2(x-1)(x+1)(2x+3) = 0.$$

$$\iff x = 1,\ x = -1,\ x = -3/2.$$ critical points of f.

Observe that $f'(x)$ is defined everywhere, so f doesn't have any singular points.

$$f(1) = 1 + \cancel{2} - \cancel{2} - 6 + 1 = -4$$

$$f(-1) = \cancel{1} - \cancel{2} - 2 + 6 + \cancel{1} = 4$$

$$f\left(-\frac{3}{2}\right) = \frac{-317}{16}.$$

	$-3/2$		-1		1	
f'	$-$		$+$		$-$	$+$
f	$\searrow$	local min.	$\nearrow$	local max	$\searrow$ local min.	$\nearrow$

Recall

Thm: Existence of Extreme Values on Open Intervals.

If f is a continuous function on the open interval (a,b) and if $\lim_{x\to a^+} f(x) = L$ & $\lim_{x\to b^-} f(x) = M$

then the followings hold:

(i) If $f(u) > M$ and $f(u) > L$ for some $u \in (a,b)$ then f has absolute maximum value on (a,b).

(ii) If $f(v) < M$ & $f(v) < L$ for some $v \in (a,b)$ then f has absolute minimum on (a,b).

Observe that f is cont.

$\mathbb{R} = (-\infty, \infty)$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^4 + 2x^3 - 2x^2 - 6x + 1) = \infty.$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^4 + 2x^3 - 2x^2 - 6x + 1) = \infty.$$

Observe that f cannot have absolute maximum on $\mathbb{R}$.

Also observe that there is some $v \in (-\infty, \infty)$ (pick any $v \in \mathbb{R}$) s.t $f(v) < \infty (= M, L)$. Therefore, by Thm, f has absolute minimum. Also, since f realizes its abs min. value at local minimum values, f has abs min value $\frac{-317}{16}$ at the pt $x = -\frac{3}{2}$.

(b) $f(x) = \arctan(x - \sqrt{x})$ on $\underline{\underline{[0, 9]}}$.

<u>Thm</u>: If f is a continuous function on a closed interval, f has absolute maximum and absolute minimum.

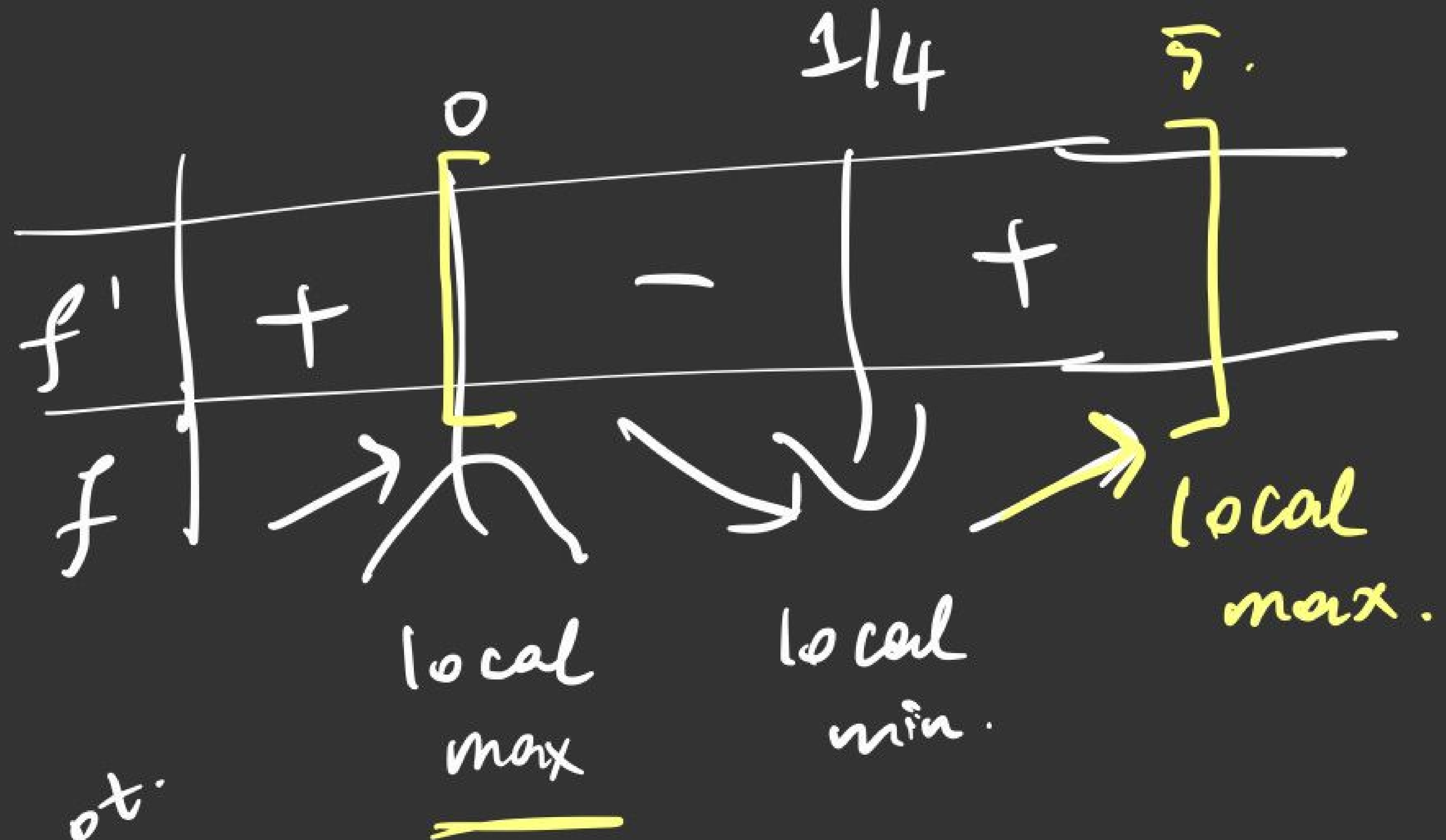
$$f'(x) = \frac{1}{1+(x-\sqrt{x})^2} \cdot (x-\sqrt{x})' = \frac{1}{1+(x-\sqrt{x})^2} \cdot \left(1 - \frac{1}{2\sqrt{x}}\right).$$

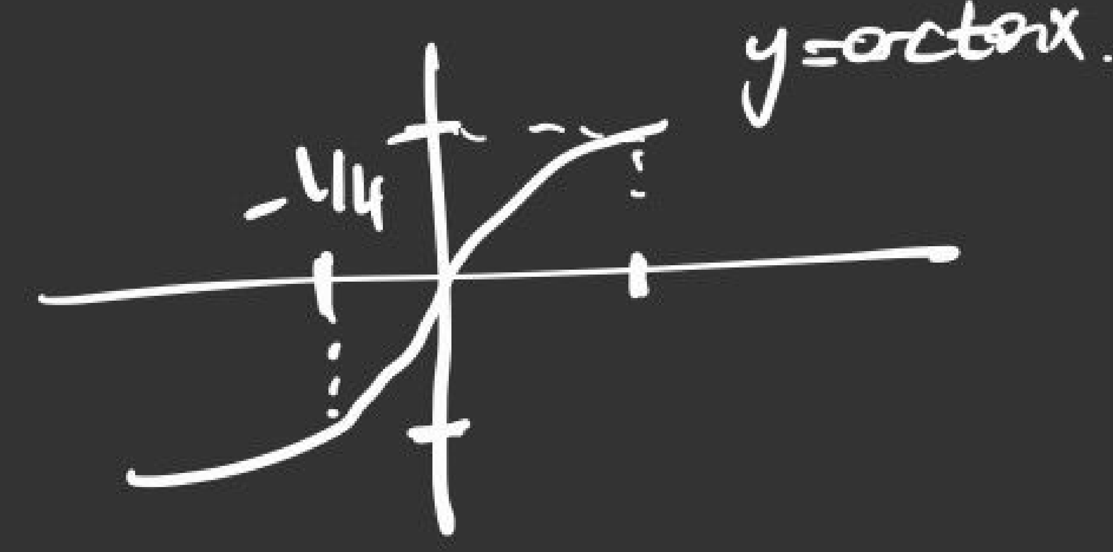
Observe that $x = 0$ is a singular point.

$$f'(x) = 0 \iff \underbrace{\frac{1}{1+\underbrace{(x-\sqrt{x})^2}_{\geq 0}}}_{\neq 0} \cdot \left(1 - \frac{1}{2\sqrt{x}}\right) = 0.$$

$$\iff 1 - \frac{1}{2\sqrt{x}} = 0 \iff \sqrt{x} = \frac{1}{2} \iff x = \frac{1}{4}$$

critical pt of f.





sing. pt. $f(0) = \arctan(0) = 0.$

crit. pt. $f(\frac{1}{4}) = \arctan(\frac{1}{4} - \frac{1}{2}) = \arctan(-\frac{1}{4}) < 0.$

end pt. $f(0) = 0.$

$f(5) = \arctan(5 - \sqrt{5}) > 0.$

Therefore, at $x=5$ f has absolute max value $\arctan(5-\sqrt{5})$ and at $x=\frac{1}{4}$ f has abs. min. value $\arctan(-\frac{1}{4})$.

#2: Find the intervals of concavity of $f(x) = x^2 \cdot e^{-x^2}$ and locate any inflection points.

inflection point: $x = x_0$ is an inflection point if

(i) $y = f(x)$ has a tangent line at $x = x_0$. $\Leftrightarrow$ $f'(x)$ exists OR $\mp\infty$.

(ii) the concavity of f changes at $x = x_0$. $\Leftrightarrow$ $f''(x_0) = 0$.

Thm: (i) If $f''(x) > 0$ on the interval I then f is concave up on I.

(ii) If $f''(x) < 0$ on I then f is concave down on I.

$$f'(x) = 2x \cdot e^{-x^2} + x^2 \cdot e^{-x^2} \cdot (-2x)$$
$$= e^{-x^2}(2x - 2x^3)$$

Note: Observe that f' exists for every $x \in \mathbb{R}$.

$$f''(x) = e^{-x^2} \cdot (-2x) \cdot (2x - 2x^3) + e^{-x^2} \cdot (2 - 6x^2)$$
$$= e^{-x^2}(4x^4 - 10x^2 + 2).$$

$$f''(x) = 0 \Leftrightarrow \underbrace{e^{-x^2}}_{\neq 0} \cdot (4x^4 - 10x^2 + 2) = 0 \Leftrightarrow$$

$$4x^4 - 10x^2 + 2 = 0 \underset{x^2 = t}{\Leftrightarrow} 4t^2 - 10t + 2 = 0$$

$\Longleftrightarrow \quad t_{1,2} = \dfrac{5 \mp \sqrt{17}}{4} \quad \Longleftrightarrow \quad x = \mp\sqrt{t}$

$\Delta = b^2 - 4ac$

$t_{1,2} = \dfrac{-b \mp \sqrt{\Delta}}{2a}$

$$x_1 = \frac{\sqrt{5+\sqrt{17}}}{2}, \quad x_2 = \frac{-\sqrt{5+\sqrt{17}}}{2}, \quad x_3 = \frac{\sqrt{5-\sqrt{17}}}{2}, \quad x_4 = \frac{-\sqrt{5-\sqrt{17}}}{2}.$$

(ii) $f''(x_i) = 0$. $\quad i = 1, \dots, 4$

(i) By the note, $f'(x_i)$ exists $(i = 1, \dots, 4)$

Hence x_i $(i = 1, \dots, 4)$ are inflection points.

		$-\frac{\sqrt{5+\sqrt{17}}}{2}$		$-\frac{\sqrt{5-\sqrt{17}}}{2}$		$\frac{\sqrt{5-\sqrt{17}}}{2}$		$\frac{\sqrt{5+\sqrt{17}}}{2}$	
f''	$+$		$-$		$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$		$\cap$		$\cup$

So f is concave up on the intervals $\left(-\infty, -\frac{\sqrt{5+\sqrt{17}}}{2}\right)$, $\left(-\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{\sqrt{5-\sqrt{17}}}{2}\right)$, $\left(\frac{\sqrt{5+\sqrt{17}}}{2}, \infty\right)$.

f is concave down on the intervals

$$\left(-\frac{\sqrt{5+\sqrt{17}}}{2}, -\frac{\sqrt{5-\sqrt{17}}}{2}\right), \left(\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{\sqrt{5+\sqrt{17}}}{2}\right).$$

#3: Find all asymptotes of $f(x) = \frac{\sqrt{2x+1}}{3x+1}$.

Recall:

vertical asymptote: $y = f(x)$ has vertical asymptote at $x = a$ if $\lim_{x \to a^-} f(x) = \mp\infty$ or $\lim_{x \to a^+} f(x) = \mp\infty$ or both.

Horizontal asymptote: $y = f(x)$ has a horizontal asymptote at $y = L$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$ or both.

Oblique asymptote: The straigth line $y = ax + b$ $(a \neq 0)$ is an oblique asymptote if either $\lim_{x \to \infty} (f(x) - (ax+b)) = 0$ or $\lim_{x \to -\infty} (f(x) - (ax+b)) = 0$ or both.

Observe that $\lim_{x \to -\frac{1}{3}} f(x) = \lim_{x \to -\frac{1}{3}} \frac{\sqrt{2x+1}}{3x+1} = \infty$.

Therefore at $x = -\frac{1}{3}$, $y = f(x)$ has a vertical asymptote.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{2x+1}}{3x+1} \overset{(\infty/\infty)}{\underset{L'H}{=}}$$

$$\lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{2x+1}}\cdot\cancel{2}}{3} = \lim_{x\to\infty} \frac{1}{3\sqrt{2x+1}} = 0 \quad (=L).$$

Therefore, $y=0$ is a horizontal asymptote for $y=f(x)$.

$$\lim_{x\to\infty} \left(f(x) - (ax+b)\right) = \lim_{x\to\infty} \left(\frac{\sqrt{2x+1}}{3x+1} - (ax+b)\right)$$

$$= \underbrace{\lim_{x\to\infty} \frac{\sqrt{2x+1}}{3x+1}}_{=0} - \underbrace{\lim_{x\to\infty} ax+b}_{\infty} = -\infty. \ (\neq 0).$$

Therefore, $y=f(x)$ has no oblique asymptote.

#4: Let f be a continuous function and its derivatives exist everywhere. Also let $f(-1)=0$

$f(0)=2$, $f(1)=1$, $f(2)=0$, $f(3)=1$ and

$$\lim_{x \to \mp\infty} (f(x) + \underbrace{1-x}_{-(x-1)}) = 0.$$

Assume that $f'(x) > 0$ on $(-\infty,-1)$, $(-1,0)$ & $(2,\infty)$ and $f'(x) < 0$ on $(0,2)$ such that

$$\lim_{x \to -1} f'(x) = \infty.$$

Suppose also that $f''(x) > 0$ on $(-\infty,-1)$ & $(1,3)$ and $f''(x) < 0$ on $(-1,1)$ & $(3,\infty)$.

Sketch the graph of f and identify the critical points.

Observations:

(1) $y = x-1$ is an oblique asymptote for $f(x) = y$.

(2). $x=-1$ is a singular point.

(3) For $x<0$, $f'(x) > 0$. $\xLeftrightarrow[\text{since } f'' \text{ exists}]{f' \text{ is cont.}}$ For $x=0$, $f'(x)=0$.

For $x>0$, $f'(x) < 0$. $f'(0) = 0$.

Therefore, $x=0$ is a critical point.

(4). For $x<2$, $f'(x)<0$
For $x>2$, $f'(x)>0$
$\xrightarrow{f' \text{ is cont}}$ (f'' exists)
For $x=2$
$f'(x)=0$.
$\Leftrightarrow f'(2)=0$.

Therefore $x=2$ is a critical point.

(5)

		-1		0		2	
f'	$+$		$+$		$-$		$+$
f	$\nearrow$		$\nearrow$		$\searrow$		$\nearrow$
				local max		local min.	

(6) For $x<-1$, $f''(x)>0$
For $x>-1$, $f''(x)<0$
$\xrightarrow{f'' \text{ cont.}}$ since (f''' exists)
For $x=-1$,
$f''(x)=0$
$\Leftrightarrow f''(-1)=0$.

Also, since at $x=-1$, the graph $y=f(x)$ has a vertical tangent line, $x=-1$ is an inflection point.

(7) For $x>1$, $f''(x)>0$
For $x<1$, $f''(x)<0$
$\xrightarrow{f'' \text{ is cont}}$ since f''' exists.
For $x=1$
$f''(x)=0$
$\Leftrightarrow f''(1)=0$.

Also, since $f'(1)$ exists, therefore $x=1$ is an inflection pt.

(8) <u>Exercise</u>: Observe that $x=3$ is also an inflection pt.