REC-VI

#1: Evaluate y' for the expression

(a) $y = (\tan x)^{\arctan x}$.

$$\left[\left(f(x)^{g(x)}\right)' = \left(e^{\ln f(x)^{g(x)}}\right)' = \left(e^{g(x).\ln(f(x))}\right)' \right]$$

$$y = e^{\ln\left((\tan x)^{\arctan x}\right)}$$

- $\arctan x = \frac{1}{1+x^2}$
- $\ln f(x) = \frac{f'(x)}{f(x)}$

$$y = e^{[\arctan x.\ln(\tan x)]}$$

$$y' = \left(e^{\arctan x.\ln(\tan x)}\right)' =$$

$$= e^{\arctan x.\ln(\tan x)} \cdot (\arctan x.\ln(\tan x))'$$

$$= \underbrace{e^{\arctan x.\ln(\tan x)}}_{y} \cdot \left(\frac{1}{1+x^2}.\ln(\tan x) + \arctan x.\frac{\sec^2 x}{\tan x}\right)$$

(b) $\ln(x+y) = \arctan(xy)$. $\quad \frac{dy}{dx} = ?$

$$\frac{d}{dx}\ln(x+y) = \frac{d}{dx}\arctan(xy).$$

$$\frac{1}{x+y}\frac{d}{dx}(x+y) = \frac{1}{1+(xy)^2}\cdot\frac{d}{dx}(xy).$$

$$\frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right) = \frac{1}{1+(xy)^2}\left(1.y + x.\frac{dy}{dx}\right).$$

$$\frac{1}{x+y} + \frac{dy}{dx}\cdot\frac{1}{x+y} = \frac{y}{1+(xy)^2} + \frac{dy}{dx}\cdot\frac{x}{1+(xy)^2}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{x+y} - \frac{x}{1+(xy)^2}\right) = \frac{y}{1+(xy)^2} - \frac{1}{x+y}.$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\frac{y}{1+(xy)^2} - \frac{1}{x+y}}{\frac{1}{x+y} - \frac{x}{1+(xy)^2}}}$$

$\bullet\ (a^x)' = \ln a \cdot a^x$

(c) $y = 2^{\arcsin(x^3)}$

$$y' = \ln 2 \cdot 2^{\arcsin(x^3)} \cdot (\arcsin(x^3))'$$

$$= \ln 2 \cdot 2^{\arcsin(x^3)} \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2.$$

$\bullet\ \left[(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}\right]$

#2: Evaluate the following limits without using L'Hopital's Rule.

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 1} \frac{1}{x-1} \cdot \ln x = \lim_{x \to 1} \ln\left(x^{\frac{1}{x-1}}\right)$

$= \ln\left(\underbrace{\lim_{x\to 1} x^{\frac{1}{x-1}}}\right) \overset{*}{=} \ln(e) = 1$

↓
$\ln x$ is cont.

$*$ $\lim_{x \to 1} x^{\frac{1}{x-1}} = \lim_{y\to 0} (1+y)^{\frac{1}{y}} \overset{*}{=} e.$

$x-1 = y$
↓
1

Proof of $*$: $(1+x)^n = 1 + n \cdot x + \frac{n \cdot (n-1)}{2!} x^2 + \frac{n \cdot (n-1)(n-2)}{3!} x^3 + \ldots$

$(1+x)^{1/x} = 1 + \frac{1}{x} \cdot x + \frac{\frac{1}{x} \cdot \left(\frac{1}{x} - 1\right)}{2!} x^2 + \frac{\frac{1}{x} \cdot \left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)}{3!} x^3 + \ldots$

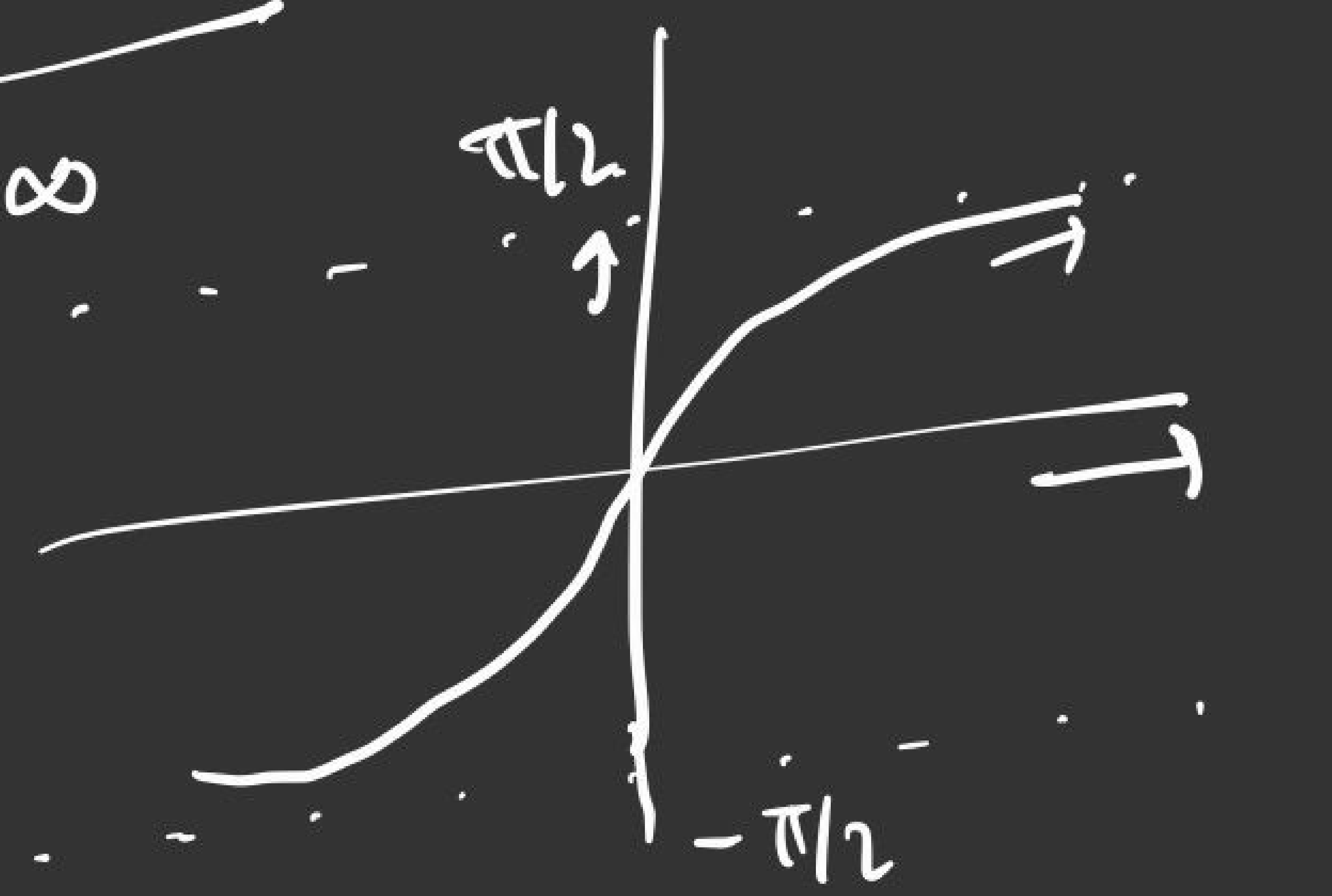
$= 1 + 1 + \frac{1 \cdot (1-x)}{2! \cancel{x^2}} \cancel{x^2} + \frac{1(1-x)(1-2x)}{3! \cancel{x^3}} \cancel{x^3} + \ldots$

$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} 1 + 1 + \frac{1(1-x)}{2!} + \frac{1 \cdot (1-x)(1-2x)}{3!} + \ldots$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

(b) $\lim_{x\to\infty} \arctan(x^3-5x+7)$.

$$\underset{\substack{\downarrow \\ \text{arctan}x \text{ is continuous.}}}{=} \arctan\Big(\underbrace{\lim_{x\to\infty} x^3-5x+7}_{\infty}\Big) = \arctan\infty = \frac{\pi}{2}.$$



(c) $\lim_{x\to 0} \frac{e^x-1}{x}$

$e^x - 1 = t.$

$e^x = 1+t$

$x = \ln(1+t)$

$$= \lim_{t\to 0} \frac{t}{\ln(1+t)} = \lim_{t\to 0} \frac{1}{\frac{\ln(1+t)}{t}} = \frac{1}{\lim_{t\to 0} \frac{\ln(1+t)}{t}}$$

since ln is a cont func.

$$= \frac{1}{\lim_{t\to 0} \ln\left((1+t)^{1/t}\right)} = \frac{1}{\ln\left(\lim_{t\to 0} (1+t)^{1/t}\right)} \overset{(*)}{=} \frac{1}{\ln(e)} = 1.$$

#3a: A sphere of ice of radius 7 cm is melting at the rate of 12 cm^3 per minute. What is the rate at which its radius is changing when the radius is 5 cm?

$$\left.\frac{dr}{dt}\right|_{r=5} = ?$$

$$\frac{dV}{dt} = -12 \text{ cm}^3/\text{min}.$$

$$V = \frac{4}{3}\pi r^3. \quad \Rightarrow \frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=5} = \left.\frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}\right|_{r=5}.$$

$$\overset{-3}{\cancel{-12}} = \frac{\cancel{4}}{\cancel{3}}\pi \cdot \cancel{3} \cdot 25 \cdot \left.\frac{dr}{dt}\right|_{r=5}.$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{-3}{25\pi} \text{ cm/min}.}$$

#3b: If a truck factory employs x workers and has daily operating expenses of y dollars, it can produce $P = \frac{x^{0.6} y^{0.4}}{3}$ trucks per year. How fast are the daily expenses decreasing when the expenses are 10000 dollars and the number of workers is 40, if the number of workers is increasing at the rate of 1 per day and the production is remaining constant?

$\frac{dy}{dt} = ?$ $y = 10000$, $x = 40$, $\frac{dx}{dt} = 1$. $\frac{dP}{dt} = 0$.

$$P = \frac{1}{3} \cdot x^{0.6} \cdot y^{0.4}$$

$$\frac{d}{dt}(P) = \frac{d}{dt}\left(\frac{1}{3} \cdot x^{0.6} \cdot y^{0.4}\right)$$

$$\Rightarrow \frac{dP}{dt} = \frac{0.6}{3} \cdot x^{-0.4} \frac{dx}{dt} \cdot y^{0.4} + \frac{0.4}{3} x^{0.6} \cdot y^{-0.6} \cdot \frac{dy}{dt}.$$

$$0 = \underline{\frac{0.6}{3} \cdot (40)^{-0.4} \cdot 1 \cdot (10000)^{0.4}} + \frac{0.4}{3} \cdot (40)^{0.6} \cdot (10000)^{-0.6} \cdot \frac{dy}{dt}.$$

$$-\frac{0.6}{3}\cdot(40)^{-0.4}\cdot(10000)^{0.4}=\frac{0.4}{3}(40)^{0.6}\cdot(10000)^{-0.6}\cdot\frac{dy}{dt}$$

$$\frac{dy}{dt}=\frac{-\frac{0.6}{\cancel{3}}\cdot(40)^{-0.4}\cdot(10000)^{0.4}}{\frac{0.4}{\cancel{3}}\cdot(40)^{0.6}\cdot(10000)^{-0.6}}$$

$$\frac{dy}{dt}=-\frac{3}{2}\cdot(40)^{-1}(10000)^{1}=-\frac{3}{2}\cdot\frac{\overset{250}{\cancel{10000}}}{\cancel{40}}$$

$$\boxed{=-375}$$

#4: Evaluate the following limits (if any)

(a) $\lim\limits_{x\to 1}\dfrac{x^{7/3}-1}{x^{5/11}-1}\;\left(\frac{0}{0}\right)\underset{L'H}{=}\lim\limits_{x\to 1}\dfrac{7/3\cdot x^{4/3}}{5/11\cdot x^{-6/11}}$

$$=\lim_{x\to 1}\frac{7}{3}\cdot\frac{11}{5}\cdot x^{4/3}\cdot x^{6/11}=\frac{77}{15}.$$

(b) $\lim_{x\to 5} \dfrac{\ln(11-2x)}{x^3-125}$ $\underset{L'H}{\overset{\left(\frac{0}{0}\right)}{=}}$

$$\lim_{x\to 5} \frac{\frac{1}{11-2x}(-2)}{3x^2} = \lim_{x\to 5}\left(\frac{-2}{3}\right)\cdot\frac{1}{(11-2x)x^2}$$

$$= -\frac{2}{3}\cdot\frac{1}{25} = \underline{\underline{\frac{-2}{75}}}.$$

(c) $\lim_{x\to 0} (\cos x)^{1/x^2}$

$$= \lim_{x\to 0} e^{\ln\left((\cos x)^{1/x^2}\right)} = \lim_{x\to 0} e^{\frac{1}{x^2}\cdot\ln(\cos x)}$$

$$= \lim_{x\to 0} e^{\frac{\ln(\cos x)}{x^2}} \underset{\substack{\downarrow\\ \text{since } e^x\\ \text{is cont.}}}{=} e^{\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}} \underset{\left(\frac{0}{0}\right)}{\overset{L'H}{=}}$$

$$e^{\lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x}} = e^{\lim_{x\to 0}\frac{-\sin x}{2x\cdot\cos x}} \underset{L'H}{\overset{\left(\frac{0}{0}\right)}{=}} e^{\lim_{x\to 0}\frac{-\cos x}{2\cos x - 2x\sin x}}$$

$$= e^{\frac{-1}{2}} = \underline{\underline{\frac{1}{\sqrt{e}}}}.$$

(d) $\lim_{x\to 3} \dfrac{\arctan(x-2) - \pi/4}{x^3-27} \underset{L'H}{\overset{\left(\frac{0}{0}\right)}{=}}$

$$= \lim_{x\to 3} \frac{\frac{1}{1+(x-2)^2}}{3x^2} = \frac{1}{2}\cdot\frac{1}{27} = \underline{\underline{\frac{1}{54}}}.$$

(e) $\lim_{x\to\infty} \dfrac{2^x+3^x}{4^x+5^x}$.

Observe:

$$\lim_{x\to\infty} \frac{2^x+3^x}{4^x+5^x} \underset{L'H}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{2^x\cdot\ln 2+3^x\cdot\ln 3}{4^x\ln 4+5^x\cdot\ln 5}$$

$$\underset{L'H}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{2^x(\ln 2)^2+3^x\cdot(\ln 3)^2}{4^x(\ln 4)^2+5^x(\ln 5)^2} \underset{L'H}{\overset{\left(\frac{\infty}{\infty}\right)}{=}}.$$

n'th L'H $= \lim_{x\to\infty} \dfrac{2^x\cdot(\ln 2)^n+3^x\cdot(\ln 3)^n}{4^x(\ln 4)^n+5^x(\ln 5)^n} \overset{\left(\frac{\infty}{\infty}\right)}{=}$.

$$\lim_{x\to\infty} \frac{2^x+3^x}{4^x+5^x} = \lim_{x\to\infty} \frac{\cancel{5^x}\left(\frac{2^x}{5^x}+\frac{3^x}{5^x}\right)}{\cancel{5^x}\left(\frac{4^x}{5^x}+1\right)}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{2}{5}\right)^x+\left(\frac{3}{5}\right)^x}{\left(\frac{4}{5}\right)^x+1} = \underline{\underline{0}}.$$