REC - VI

$\ln x^r = r.\ln x.$

#1: Evaluate y' for the expression

(a) $y = (\tan x)^{\arctan x}.$

$$\left[\begin{aligned} y &= (f(x))^{g(x)} \\ y &= e^{\ln y} = e^{\ln((f(x))^{g(x)})} \\ &= e^{g(x).\ln(f(x))} \end{aligned}\right]$$

$$y = e^{\ln\left((\tan x)^{\arctan x}\right)}$$

$$= e^{\arctan x . \ln(\tan x)}$$

$$y' = e^{\arctan x . \ln(\tan x)} . \left(\arctan x . \ln(\tan x)\right)'$$

$$= \underbrace{e^{\arctan x . \ln(\tan x)}}_{=y} . \left(\frac{1}{1+x^2} . \ln(\tan x) + \arctan x . \frac{\sec^2 x}{\tan x}\right).$$

(b) $\ln(x+y) = \arctan(xy).$

$$\frac{d}{dx}\left(\ln(x+y)\right) = \frac{d}{dx}\left(\arctan(xy)\right).$$

$$\frac{1}{x+y} \frac{d}{dx}(x+y) = \frac{1}{1+(xy)^2} . \frac{d}{dx}(xy).$$

$$\frac{1}{x+y} . \left(1 + \frac{dy}{dx}\right) = \frac{1}{1+(xy)^2} . \left(1.y + x.\frac{dy}{dx}\right).$$

$$\frac{1}{x+y} + \frac{1}{x+y}\frac{dy}{dx} = \frac{y}{1+(xy)^2} + \frac{x}{1+(xy)^2}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{x+y} - \frac{x}{1+(xy)^2}\right) = \frac{y}{1+(xy)^2} - \frac{1}{x+y}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{y}{1+(xy)^2} - \frac{1}{x+y}}{\frac{1}{x+y} - \frac{x}{1+(xy)^2}}}$$

- $(a^x)' = \ln a \cdot a^x$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

(c) $y = 2^{\arcsin(x^3)}$

$$y' = \ln 2 \cdot 2^{\arcsin(x^3)} \cdot (\arcsin(x^3))'$$

$$= \ln 2 \; 2^{\arcsin(x^3)} \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2$$

#2: Evaluate the following limits without using L'Hopital's Rule.

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$.

$$= \lim_{x \to 1} \frac{1}{x-1} \cdot \ln x = \lim_{x \to 1} \ln \left(x^{\frac{1}{x-1}} \right).$$

$$= \ln \left(\lim_{x \to 1} x^{\frac{1}{x-1}} \right) = \ln(e) = \underline{\underline{1}}.$$

* since $\ln x$ is cont.

$$\lim_{x \to 1} x^{\frac{1}{x-1}} = \lim_{\substack{x-1=y \\ y \to 0}} (1+y)^{1/y} = e. \quad (*)$$

$x = 1+y$.

Proof of (*):

$$(1+x)^n = 1 + n \cdot x + \frac{n \cdot (n-1)}{2!} x^2 + \frac{n \cdot (n-1)(n-2)}{3!} x^3 + \dots$$

$n = \frac{1}{x}$.

$$(1+x)^{1/x} = 1 + \frac{1}{x} \cdot x + \frac{\frac{1}{x} \cdot \left(\frac{1}{x} - 1\right)}{2!} x^2 + \frac{\frac{1}{x} \cdot \left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)}{3!} x^3 + \dots$$

$$= 1 + 1 + \frac{1\cdot(1-x)}{2!\,\cancel{x^2}}\cdot\cancel{x^2} + \frac{1(1-x)(1-2x)}{3!\,\cancel{x^3}}\cancel{x^3} + \cdots$$

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}\left(1+1+\frac{1\cdot(1-\overset{0}{x})}{2!} + \frac{1\cdot(1-x)(1-2x)}{3!} + \cdots\right)$$

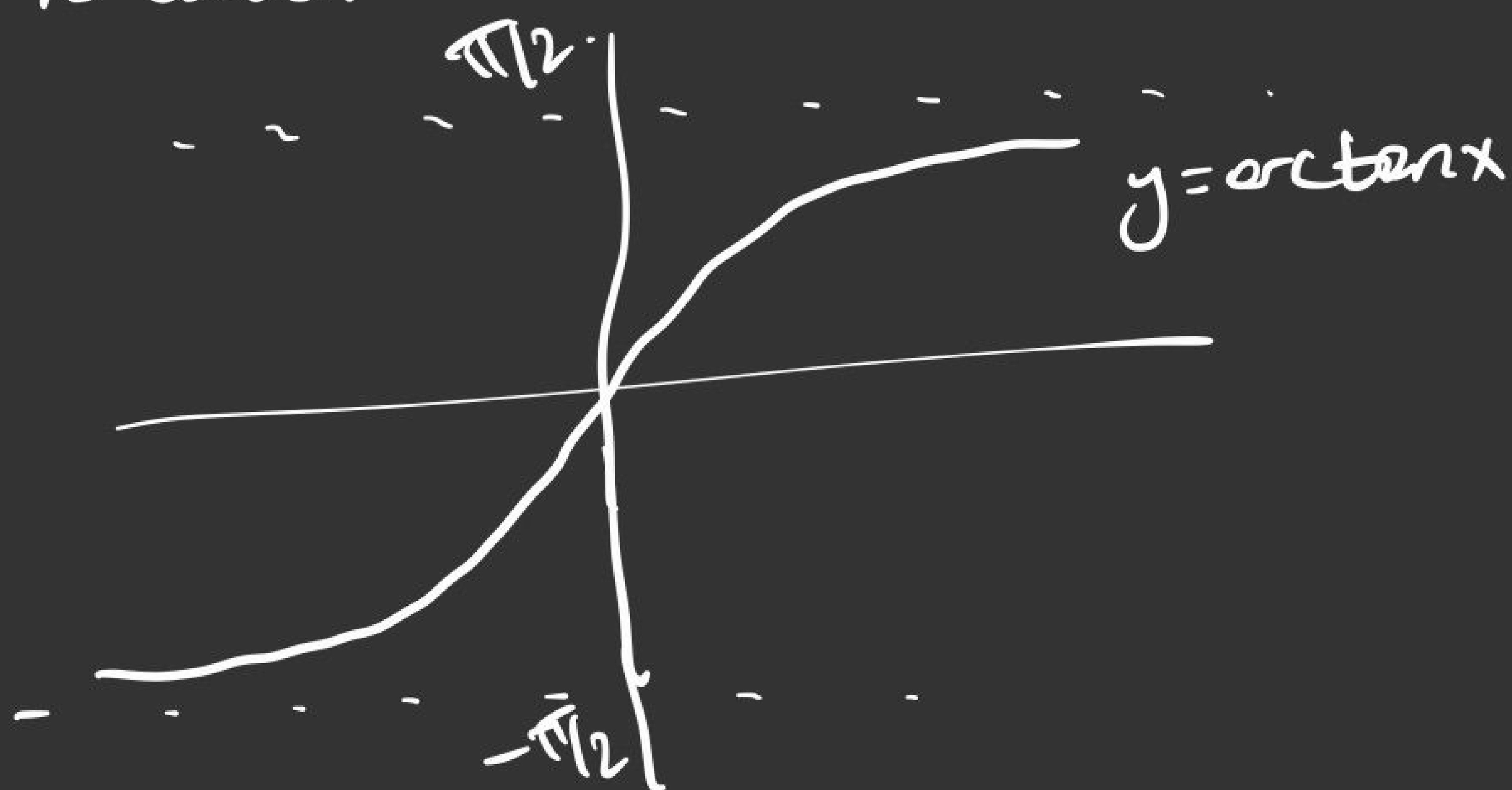
$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!} = e. \quad \blacksquare$$

(b) $\lim_{x\to\infty} \arctan(x^3 - 5x + 7)$.

$$= \arctan\Big(\underbrace{\lim_{x\to\infty} x^3 - 5x + 7}_{\infty}\Big) = \arctan\infty.$$

$$= \frac{\pi}{2}.$$

* since arctan x is cont.



(c) $\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{t \to 0} \frac{t}{\ln(1+t)}$

$e^x - 1 = t \to 0$

$e^x = 1 + t$

$x = \ln(1+t)$

$$= \lim_{t \to 0} \frac{1}{\frac{\ln(1+t)}{t}} = \frac{1}{\lim_{t \to 0} \frac{\ln(1+t)}{t}}$$

$$= \frac{1}{\lim_{t \to 0} \frac{1}{t} \cdot \ln(1+t)} = \frac{1}{\lim_{t \to 0} \ln\left((1+t)^{1/t}\right)}$$

$$= \frac{1}{\ln\left(\lim_{t \to 0} (1+t)^{1/t}\right)} = \frac{1}{\ln e} = \underline{\underline{1}}$$

* since $\ln x$ is cont.

#3a: A sphere of ice of radius 7 cm is melting at the rate of 12 cm^3 per minute. What is the rate at which its radius is changing when the radius is 5 cm?

$$\left.\frac{dr}{dt}\right|_{r=5} = ?$$

$$\frac{dV}{dt} = -12 \text{ cm}^3/\text{min}.$$

$$\boxed{V = \frac{4}{3}\pi r^3.}$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=5} = \left.\frac{4}{3}\pi\, 3r^2 \cdot \frac{dr}{dt}\right|_{r=5}.$$

$$\underset{-3.}{\cancel{-12}} = \frac{\cancel{4}}{\cancel{3}}\pi \cdot \cancel{3} \cdot 25 \cdot \left.\frac{dr}{dt}\right|_{r=5}.$$

$$\Rightarrow \left.\frac{dr}{dt}\right|_{r=5} = \frac{-3}{25\pi} \text{ cm/min}.$$

#36: If a truck factory employs x workers and has daily operating expenses of y dollars, it can produce $P = \frac{x^{0.6} \cdot y^{0.4}}{3}$ trucks per year.

How fast are the daily expenses decreasing when the expenses are 10000 dollars and the number of workers is 40, if the number of workers increasing at the rate of 1 per day and the production is remaining constant?

$$\frac{dy}{dt} = ? \qquad y = 10000, \quad x = 40, \quad \frac{dx}{dt} = 1, \quad \frac{dP}{dt} = 0.$$

$$\frac{d}{dt}(P) = \frac{d}{dt}\left(\frac{1}{3} \cdot x^{0.6} \cdot y^{0.4}\right).$$

$$\frac{dP}{dt} = \frac{0.6}{3} \cdot x^{-0.4} \cdot \frac{dx}{dt} \cdot y^{0.4} + \frac{0.4}{3} \cdot x^{0.6} \cdot y^{-0.6} \cdot \frac{dy}{dt}.$$

$$0 = \frac{0.6}{3} \cdot (40)^{-0.4} \cdot 1 \cdot (10000)^{0.4} + \frac{0.4}{3} \cdot (40)^{0.6} \cdot (10000)^{-0.6} \cdot \frac{dy}{dt}.$$

$$\frac{dy}{dt} = \frac{-\frac{0.6}{\cancel{3}} (40)^{-0.4} \cdot (10000)^{0.4}}{\frac{0.4}{\cancel{3}} \cdot (40)^{0.6} \cdot (10000)^{-0.6}}.$$

$$= -\frac{3}{2} \cdot (40)^{-1} \cdot (10000)^{1} = -\frac{3}{2} \cdot \frac{\cancel{10000}^{250}}{\cancel{40}} = -375$$

#4: Evaluate the following limits (if any)

(a) $$\lim_{x \to 1} \frac{x^{7/3} - 1}{x^{5/11} - 1} \overset{\left(\frac{0}{0}\right)}{\underset{L'H.}{=}} \lim_{x \to 1} \frac{7/3 \cdot x^{4/3}}{5/11 \cdot x^{-6/11}}$$

$$= \frac{7}{3} \cdot \frac{11}{5} \cdot \underbrace{\lim_{x \to 1} x^{4/3} \cdot x^{6/11}}_{=1} = \frac{77}{15}$$

(b) $$\lim_{x \to 5} \frac{\ln(11 - 2x)}{x^3 - 125} \overset{\left(\frac{0}{0}\right)}{\underset{L'H.}{=}} \lim_{x \to 5} \frac{\frac{1}{11 - 2x}(-2)}{3x^2}$$

$$= \lim_{x \to 5} \frac{-2}{3} \cdot \frac{1}{(11 - 2x) x^2} = \frac{-2}{75}$$

(c) $\lim\limits_{x \to 0} (\cos x)^{1/x^2}$

$$= \lim_{x \to 0} e^{\ln\left((\cos x)^{1/x^2}\right)} = \lim_{x \to 0} e^{\frac{1}{x^2} \cdot \ln(\cos x)}$$

$$= e^{\lim\limits_{x \to 0} \frac{\ln(\cos x)}{x^2}} \quad \left(\frac{0}{0}\right) \underset{L'H}{=} e^{\lim\limits_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{2x}}$$

* Since e^x is cont.

$$= e^{\lim\limits_{x \to 0} \frac{-\sin x}{\cos x \cdot 2x}} \quad \left(\frac{0}{0}\right) \underset{L'H}{=} e^{\lim\limits_{x \to 0} \frac{\overset{-1}{-\cos x}}{\underbrace{(-\sin x) \cdot 2x}_{0} + \underbrace{2\cos x}_{2}}}$$

$$= e^{-1/2} = \frac{1}{\sqrt{e}}$$

(d) $\lim\limits_{x \to 3} \dfrac{\arctan(x-2) - \pi/4}{x^3 - 27}$

$$\left(\frac{0}{0}\right) \underset{L'H}{=} \lim_{x \to 3} \frac{\frac{1}{1+(x-2)^2}}{3x^2} = \lim_{x \to 3} \frac{1}{3x^2\left(1+(x-2)^2\right)}$$

$$= \frac{1}{54}$$

(e) $\lim_{x\to\infty} \dfrac{2^x+3^x}{4^x+5^x}$. $\qquad \bullet\ (a^x)' = a^x \cdot \ln a$.

Observe:

$$\lim_{x\to\infty} \frac{2^x+3^x}{4^x+5^x} \underset{L'H}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{2^x\cdot\ln 2 + 3^x\cdot \ln 3}{4^x\cdot \ln 4 + 5^x\cdot \ln 5}$$

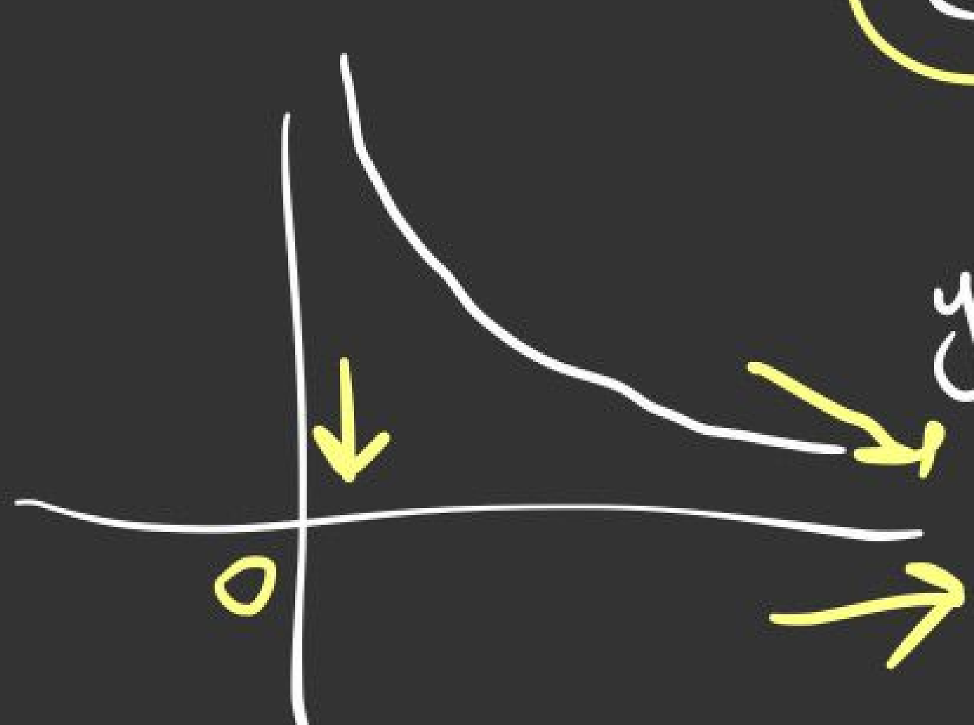
$$\underset{L'H}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{2^x\cdot(\ln 2)^2 + 3^x\cdot(\ln 3)^2}{4^x\cdot(\ln 4)^2 + 5^x\cdot(\ln 5)^2} =$$

So, if we apply L'H for n times, we will get

$$\lim_{x\to\infty} \frac{2^x(\ln 2)^n + 3^x(\ln 3)^n}{4^x(\ln 4)^n + 5^x\cdot(\ln 5)^n} \overset{\infty/\infty}{=}$$

$$\lim_{x\to\infty} \frac{2^x+3^x}{4^x+5^x} = \lim_{x\to\infty} \frac{\cancel{5^x}\left(\frac{2^x}{5^x}+\frac{3^x}{5^x}\right)}{\cancel{5^x}\left(\frac{4^x}{5^x}+1\right)}$$

$$= \lim_{x\to\infty} \frac{\underbrace{\left(\frac{2}{5}\right)^x}_{\to 0} + \underbrace{\left(\frac{3}{5}\right)^x}_{\to 0}}{\underbrace{\left(\frac{4}{5}\right)^x}_{\to 0} + 1} = 0.$$



$y=a^x$

$0<a<1$.