REC - VI

$\ln x^r = r.\ln x.$

#1: Evaluate y' for the expression

(a) $y = (\tan x)^{\arctan x}$.

$$\left[\begin{aligned} y &= f(x)^{g(x)} \\ y &= e^{\ln y} = e^{\ln\left((f(x))^{g(x)}\right)} \\ &= e^{g(x).\ln f(x)} \end{aligned}\right]$$

$$y = e^{\ln\left((\tan x)^{\arctan x}\right)}$$

$$= e^{\arctan x.\ln(\tan x)}$$

$$y' = e^{\arctan x.\ln(\tan x)} \cdot \left(\arctan x.\ln(\tan x)\right)'$$

$$= \underbrace{e^{\arctan x.\ln(\tan x)}}_{=y} \cdot \left(\frac{1}{1+x^2}.\ln(\tan x) + \arctan x.\frac{\sec^2 x}{\tan x}\right).$$

(b) $\ln(x+y) = \arctan(xy)$.

$$\frac{d}{dx}\left(\ln(x+y)\right) = \frac{d}{dx}\left(\arctan(xy)\right)$$

$$\frac{1}{x+y}\cdot\frac{d}{dx}(x+y) = \frac{1}{1+(xy)^2}\,\frac{d}{dx}(xy).$$

$$\frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right) = \frac{1}{1+(xy)^2}\cdot\left(1.y + x.\frac{dy}{dx}\right).$$

$$\frac{1}{x+y}+\frac{1}{x+y}\frac{dy}{dx}=\frac{y}{1+(xy)^2}+\frac{x}{1+(xy)^2}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{x+y}-\frac{x}{1+(xy)^2}\right)=\frac{y}{1+(xy)^2}-\frac{1}{x+y}.$$

$$\boxed{\frac{dy}{dx}=\frac{\frac{y}{1+(xy)^2}-\frac{1}{x+y}}{\frac{1}{x+y}-\frac{x}{1+(xy)^2}}}$$

(c) $y=2^{\arcsin(x^3)}$.

- $(a^x)'=\ln a.\,a^x$
- $(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}}$

$$y'=\left(2^{\arcsin(x^3)}\right)'$$

$$=\ln 2.\,2^{\arcsin(x^3)}\cdot\left(\arcsin(x^3)\right)'$$

$$=\ln 2.\,2^{\arcsin(x^3)}\cdot\frac{1}{\sqrt{1-x^6}}\cdot 3x^2$$

#2: Evaluate the following limits without using L'Hopital's Rule.

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$.

$= \lim_{x \to 1} \frac{1}{x-1} \cdot \ln x = \lim_{x \to 1} \ln\left(x^{\frac{1}{x-1}}\right) =$

$= \ln\left(\lim_{x \to 1} x^{\frac{1}{x-1}}\right) = \ln(e) = 1.$

* since $\ln x$ is continuous.

$\lim_{x \to 1} x^{\frac{1}{x-1}} = \lim_{y \to 0} (1+y)^{1/y} = e.$ ⊛

$x-1=y$

$x = 1+y$

Proof of ⊛:

$(1+x)^n = 1 + n \cdot x + \frac{n \cdot (n-1)}{2!} x^2 + \frac{n \cdot (n-1)(n-2)}{3!} x^3 + \cdots$

$n = 1/x$.

$(1+x)^{1/x} = 1 + \frac{1}{x} \cdot x + \frac{\frac{1}{x} \cdot \left(\frac{1}{x} - 1\right)}{2!} x^2 + \frac{\frac{1}{x} \cdot \left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)}{3!} x^3 + \cdots$

$$= 1 + 1 + \frac{1.(1-x)}{2!\,\cancel{x^2}}\cdot\cancel{x^2} + \frac{1(1-x)(1-2x)}{3!\,\cancel{x^3}}\cdot\cancel{x^3} + \cdots$$

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}\left[1 + 1 + \frac{1(1-x)}{2!} + \frac{1(1-x)(1-2x)}{3!} + \cdots\right]$$

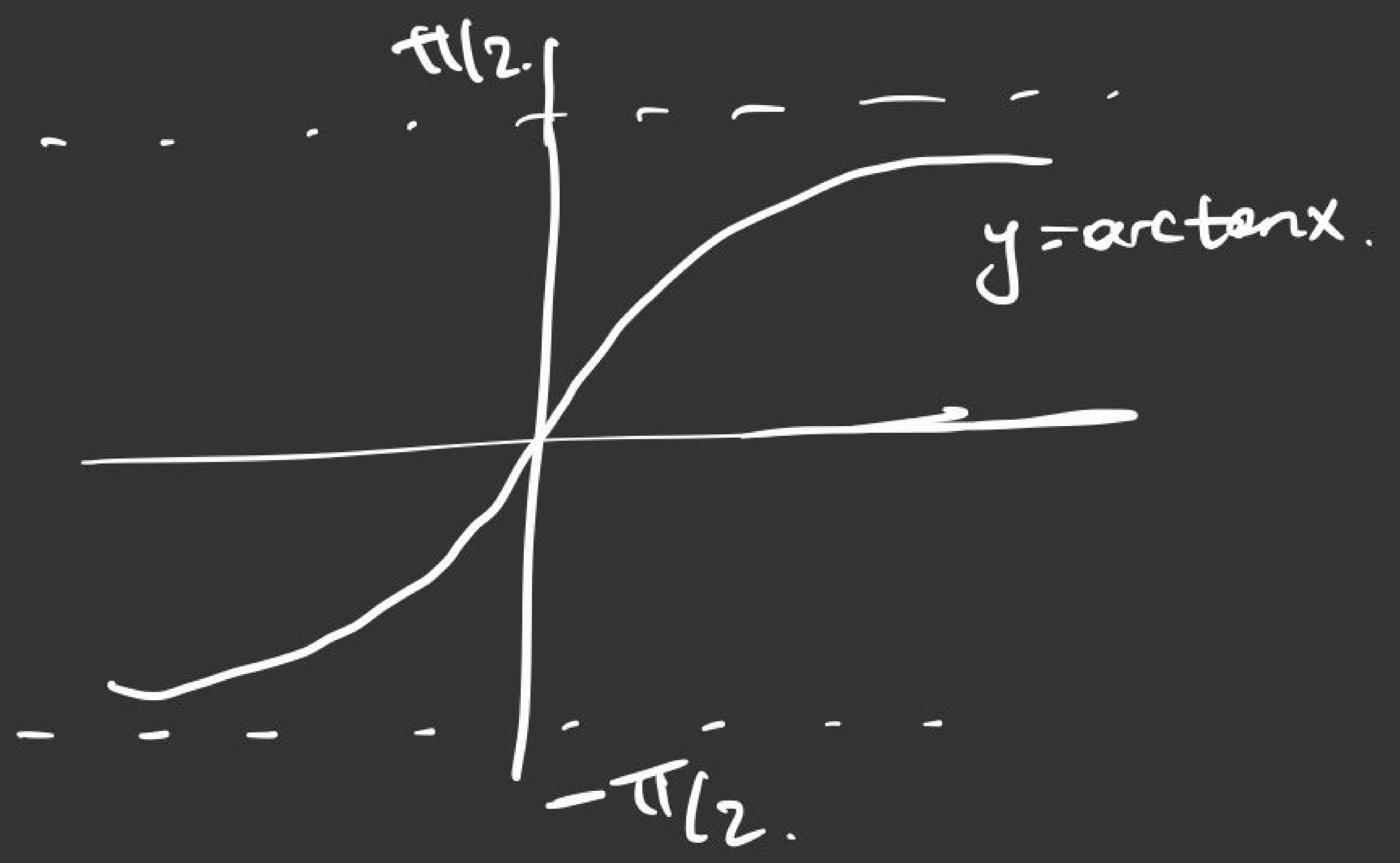
$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!} = e.$$

(b) $\lim_{x\to\infty} \arctan(x^3 - 5x + 7)$

$$\underset{\uparrow}{=} \arctan\big(\underbrace{\lim_{x\to\infty} x^3 - 5x + 7}\big)$$

since $\arctan x$ is continuous.

$$= \arctan\infty = \pi/2.$$



(c) $\lim_{x \to 0} \frac{e^x - 1}{x}$.

$e^x - 1 = t$.

$e^x = 1 + t$.

$x = \ln(1+t)$.

$$= \lim_{t \to 0} \frac{t}{\ln(1+t)}$$

$$= \lim_{t \to 0} \frac{1}{\frac{\ln(1+t)}{t}} = \frac{1}{\lim_{t \to 0} \frac{\ln(1+t)}{t}}$$

$$= \frac{1}{\lim_{t \to 0} \ln\left((1+t)^{1/t}\right)} = \frac{1}{\ln\left(\underbrace{\lim_{t \to 0} (1+t)^{1/t}}_{=e}\right)}$$

$$= \frac{1}{\ln e} = \underline{\underline{1}}.$$

#3a: A sphere of ice of radius 7 cm is melting at the rate of 12 cm^3 per minute. What is the rate at which its radius is changing when the radius is 5 cm?

$$\left.\frac{dr}{dt}\right|_{r=5} = ?$$

$$\frac{dV}{dt} = -12 \text{ cm}^3/\text{min}.$$

$$V = \frac{4}{3}\pi r^3.$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 . \frac{dr}{dt}.$$

$$\underbrace{\left.\frac{dV}{dt}\right|_{r=5}}_{-12.} = \frac{4}{3}\pi\, 3r^2 \left.\frac{dr}{dt}\right|_{r=5} = \frac{4}{3}\pi . 3 . 5^2 . \left.\frac{dr}{dt}\right|_{r=5}.$$

$$\left.\frac{dr}{dt}\right|_{r=5} = \frac{\overset{-3}{\cancel{-12}}}{\cancel{\frac{4}{3}}\pi . \cancel{3} . 5^2} = \frac{-3}{25\pi} \text{ cm/min}.$$

#3b: If a truck factory employs x workers and has daily expenses of y dollars, it can produce

$P = \frac{x^{0.6} \cdot y^{0.4}}{3}$ trucks per year. How fast are the daily expenses decreasing when the expenses are 10000 dollars and the number of workers is 40, if the number of workers is increasing at the rate of 1 per day and the production is remaining constant?

$\frac{dy}{dt} = ?$ $\quad y = 10000, \quad x = 40, \quad \frac{dx}{dt} = 1.$

$\frac{dP}{dt} = 0.$

$$\frac{d}{dt}(P) = \frac{d}{dt}\left(\frac{1}{3} x^{0.6} \cdot y^{0.4}\right)$$

$$\frac{dP}{dt} = \frac{0.6}{3} \cdot x^{-0.4} \cdot \frac{dx}{dt} \cdot y^{0.4} + \frac{0.4}{3} \cdot x^{0.6} \cdot y^{-0.6} \cdot \frac{dy}{dt}.$$

$$0 = \frac{0.6}{3} \cdot (40)^{-0.4} \cdot 1 \cdot (10000)^{0.4} + \frac{0.4}{3} (40)^{0.6} \cdot (10000)^{-0.6} \cdot \frac{dy}{dt}.$$

$$\frac{dy}{dt} = \frac{-\frac{0.6}{\cancel{3}}(40)^{-0.4}(10000)^{0.4}}{\frac{0.4}{\cancel{3}}\cdot(40)^{0.6}\cdot(10000)^{-0.6}}$$

$$= -\frac{3}{2}\cdot(40)^{-1}\cdot(10000)^{1}$$

$$= -\frac{3}{2}\cdot\frac{\cancel{10000}^{250}}{\cancel{40}} = -375.$$

#4: Evaluate the following limits (if any).

(a) $\lim_{x\to 1} \frac{x^{7/3}-1}{x^{5/11}-1} \left(\frac{0}{0}\right) \underset{L'H.}{=} \lim_{x\to 1} \frac{7/3 \cdot x^{4/3}}{5/11 \cdot x^{-6/11}}$

$$= \lim_{x\to 1} \frac{7}{3}\cdot\frac{11}{5}\cdot x^{4/3}\cdot x^{6/11} = \frac{77}{15}.$$

(b) $\lim_{x\to 5} \frac{\ln(11-2x)}{x^3-125} \cdot \left(\frac{0}{0}\right) \underset{L'H.}{=} \lim_{x\to 5} \frac{\frac{1}{11-2x}\cdot(-2)}{3x^2}$

$$= \lim_{x\to 5} -\frac{2}{3}\cdot\frac{1}{(11-2x)x^2} = \frac{-2}{75}.$$

(c) $\lim_{x \to 0} (\cos x)^{1/x^2}$

$$= \lim_{x \to 0} e^{\ln\left((\cos x)^{1/x^2}\right)}$$

$$= \lim_{x \to 0} e^{\frac{1}{x^2} \cdot \ln(\cos x)} \overset{\text{since } e^x \text{ is cont.}}{=} e^{\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}}$$

$$\overset{(\frac{0}{0})}{\underset{L'H.}{=}} e^{\lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{2x}} = e^{\lim_{x \to 0} \frac{-\sin x}{\cos x \cdot 2x}}$$

$$\overset{(\frac{0}{0})}{\underset{L'H.}{=}} e^{\lim_{x \to 0} \frac{-\cos x}{(-\sin x) \cdot 2x + \cos x \cdot 2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

(d) $\lim_{x \to 3} \frac{\arctan(x-2) - \pi/4}{x^3 - 27}$

$$\overset{(\frac{0}{0})}{\underset{L'H.}{=}} \lim_{x \to 3} \frac{\frac{1}{1+(x-2)^2}}{3x^2} = \lim_{x \to 3} \frac{1}{3x^2(1+(x-2)^2)}$$

$$= \frac{1}{54}$$

(e) $\lim_{x \to \infty} \dfrac{2^x + 3^x}{4^x + 5^x}$

Observe:

$$\lim_{x\to\infty} \frac{2^x + 3^x}{4^x+5^x} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{L'H.}{=}} \lim_{x\to\infty} \frac{2^x . \ln 2 + 3^x . \ln 3}{4^x . \ln 4 + 5^x . \ln 5}.$$

$$\overset{\left(\frac{\infty}{\infty}\right)}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{2^x . (\ln 2)^2 + 3^x (\ln 3)^2}{4^x . (\ln 4)^2 + 5^x (\ln 5)^2}.$$

If we apply L'H n times

$$\ldots = \lim_{x\to\infty} \frac{2^x (\ln 2)^n + 3^x . (\ln 3)^n}{4^x . (\ln 4)^n + 5^x . (\ln 5)^n} \overset{\frac{\infty}{\infty}}{=}.$$

$$\lim_{x\to\infty} \frac{2^x + 3^x}{4^x + 5^x} = \lim_{x\to\infty} \frac{\cancel{5^x}\left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right)}{\cancel{5^x}\left(\frac{4^x}{5^x} + 1\right)}$$

$$= \lim_{x\to\infty} \frac{\underbrace{\left(\frac{2}{5}\right)^x}_{\to 0} + \underbrace{\left(\frac{3}{5}\right)^x}_{\to 0}}{\underbrace{\left(\frac{4}{5}\right)^x}_{0} + 1} = \underline{\underline{0}}.$$

(f) $\displaystyle\lim_{x \to 0} \frac{x - \sin x}{\ln(1+7x^3)}$

$$\left(\frac{0}{0}\right) \underset{L'H.}{=} \lim_{x \to 0} \frac{1 - \cos x}{\dfrac{21x^2}{1+7x^3}} = \lim_{x \to 0} \frac{(1-\cos x)(1+7x^3)}{21x^2}$$

$$\left(\frac{0}{0}\right) \underset{L'H.}{=} \lim_{x \to 0} \frac{\sin x \cdot (1+7x^3) + (1-\cos x)\, 21x^2}{42x}.$$

$$\left(\frac{0}{0}\right) \underset{L'H}{=} \lim_{x \to 0} \frac{\cos x \cdot (1+7x^3) + \sin x \cdot 21x^2 + \sin x \cdot 21x^2 + 42x(1-\cos x)}{42}.$$

$$= \frac{1}{42}$$