REC-V

#1: Show that $x^3+x^2+3x+7=0$ has exactly one real root.

Recall: Intermediate Value Theorem

If f is cont. on $[a,b]$ and if s is a number between $f(a)$ & $f(b)$, there exists some $c \in [a,b]$ s.t. $f(c)=s$.

Mean Value Theorem

If f is cont. on $[a,b]$ and f is differentiable on (a,b), then there exists some $c \in (a,b)$ s.t $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Set $f(x) = x^3+x^2+3x+7$.

$f(0) = 7 > 0$

$f(-2) = \underbrace{-8+4}_{-4} \underbrace{-6+7}_{1} = -3 < 0$.

Observe that f is cont. everywhere.

Since f is cont. on $[-2,0]$ and $\underbrace{f(-2)}_{=-3} < 0 < \underbrace{f(0)}_{=7}$, by IVT, there exists some $c \in [-2,0]$ such that $f(c) = 0$.

Therefore, $f(x) = x^3 + x^2 + 3x + 7$ has <u>at least</u> one root.

Now, assume that f has two different roots, say c & d. Then $f(c) = 0 = f(d)$ $(c < d)$ WLOG

Observe that f is cont. on $[c,d]$ and f is diff'ble on (c,d). Therefore, by MVT, there exists some $e \in (c,d)$ such that

$$\underline{f'(e)} = \frac{f(d) - f(c)}{d - c} = \frac{0}{d-c} = \underline{0}.$$

$$f'(x) = 3x^2 + 2x + 3 \qquad (ax^2 + bx + c).$$

$$\Delta = b^2 - 4ac = 4 - 4 \cdot 3 \cdot 3 = -32 < 0.$$

Contradiction ($\lightning$), since $f'(x)$ cannot be zero, there cannot be such e.

Hence, f must have exactly one root.

#2: Prove that $e^x \geq x+1$ for all $x \in \mathbb{R}$.

we'll consider 3 cases: $x=0$, $x<0$, $x>0$.

Case 1: $x=0$

$\underbrace{e^0}_{=1} \geq \underbrace{0+1}_{=1}$ ✓ Inequality holds.

Case 2: $(x<0)$ $f(t)=e^t$. $(f'(t)=e^t)$.

f is cont & diff'ble $\forall x \in \mathbb{R}$.

Observe that
(i) f is continuous on $[x,0]$.
(ii) f is differentiable on $(x,0)$.

By MVT, there exists some $c \in (x,0)$ s.t

$$\frac{1-e^x}{-x} = \frac{e^0-e^x}{-x} = \frac{f(0)-f(x)}{0-x} = f'(c) = e^c < 1.$$

$c<0$

$$\Rightarrow \frac{1-e^x}{-x} < 1. \Rightarrow 1-e^x < -x$$

$$\Rightarrow e^x > 1+x.$$

Case 3: $(x>0)$

Observe that (i) f is cont. on $[0,x]$
(ii) f is diff'ble on $(0,x)$.

Therefore, by MVT, there exists $d \in (0,x)$ s.t.

$$\frac{e^x-1}{x} = \frac{e^x-e^0}{x} = \frac{f(x)-f(0)}{x-0} = f'(d) = e^d > 1.$$

$d > 0$

$e^d > e^0 = 1.$

$$\frac{e^x-1}{x} > 1 \Rightarrow e^x - 1 > x$$

$$\Rightarrow e^x > x+1.$$

Therefore, $e^x \geq x+1 \quad \forall x \in \mathbb{R}$.

#3: Let f be a function defined on $[0,9]$ and f', f'' exist on $(0,9)$. If $f(1)=-2$, $f(3)=5$, $f(4)=6$ and $f(8)=20$ then show that there is a number $c \in (0,9)$ s.t.

$f''(c)=0$.

[Thm: If f is differentiable at a, then f is cont. at a]

Observe

f is diff'ble on $(0,9)$. Therefore,

f is cont on $(0,9)$.

(i) f is cont $[1, 3]$ $(\subset (0,9))$

(ii) f is diff'ble $(1, 3)$ $(\subset (0,9))$

By MVT, $\exists d \in (1, 3)$ s.t

$$f'(d) = \frac{f(3) - f(1)}{3-1} = \frac{5-(-2)}{2} = \frac{7}{2}.$$

(i) f is cont on $[4, 8]$ $(\subset (0,9))$

(ii) f is diff'ble on $(4, 8)$ $(\subset (0,9))$

By MVT, $\exists e \in (4, 8)$ s.t

$$f'(e) = \frac{f(8) - f(4)}{8-4} = \frac{20-6}{4} = \frac{7}{2}$$

Now observe that since f'' exists on $(0,9)$, it means that f' is differentiable on $(0,9)$.

Therefore, f' is continuous on $(0,9)$.

(i) f' is cont on $[d, e]$ $(\subset (0,9))$

(ii) f' is diff'ble on (d, e) $(\subset (0,9))$.

Now by MVT, $\exists\, c \in (d, e)\ (C(0,9))$.

$$\underline{\underline{f''(c)}} = \frac{f'(e) - f'(d)}{e-d} = \frac{7/2 - 7/2}{e-d} = \underline{\underline{0}}$$

#4: Given $f(x) = x^5 + 7x - 2\sin(\pi x) - 2$

(a) Show that $f^{-1}(x)$ exists.

$f : \text{Dom} f \to \text{Ran} f$

Fact: ① If f is 1-1, then f^{-1} exists.

② If f is a strictly increasing function then f is 1-1.

③ f is strictly increasing if $f' > 0$.

$$f'(x) = \underbrace{5x^4}_{>0} + \underbrace{7 - 2\pi\cos(\pi x)}_{>0} > 0.$$

Since $-1 \leq \cos(\pi x) \leq 1$, we have

$$-2\pi \leq -2\pi\cos(\pi x) \leq 2\pi$$

$$0 < 7 - 2\pi \leq 7 - 2\pi\cos(\pi x) \leq 7 + 2\pi$$

Therefore, $f'(x) > 0 \ \forall x \in \text{Dom} f$, so f is strictly increasing. Therefore (from fact 2) f must be 1-1. So, f^{-1} must exist.

(b) Find the domain and range of f^{-1}.

$f(x) = x^5 + 7x - 2 - 2\sin(\pi x)$.

$f : \text{Dom} f \to \text{Ran} f \qquad f^{-1} : \text{Ran} f \to \text{Dom} f$.

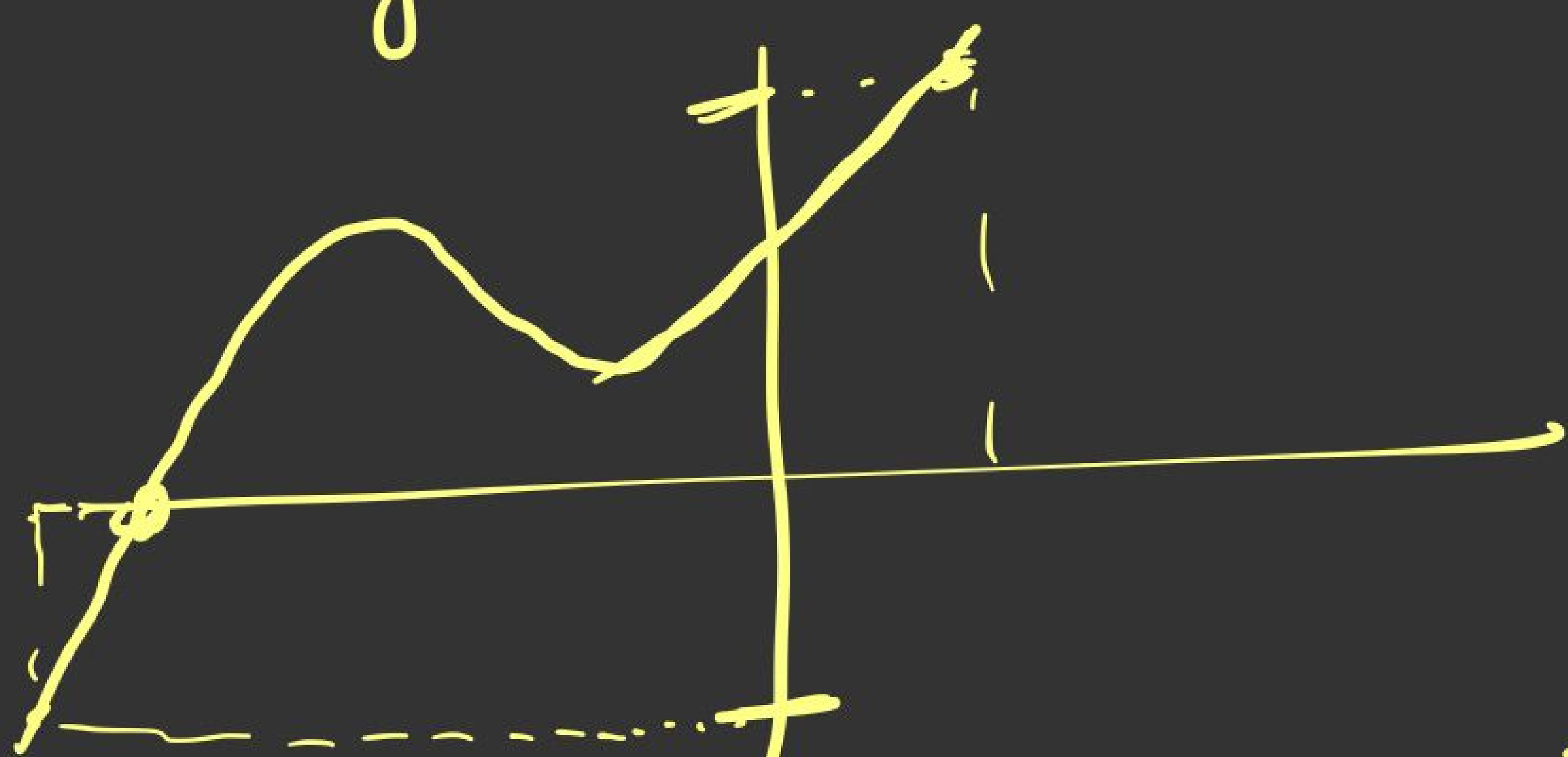
$\text{Dom} f^{-1} = \text{Ran} f$ and $\text{Ran} f^{-1} = \text{Dom} f$.

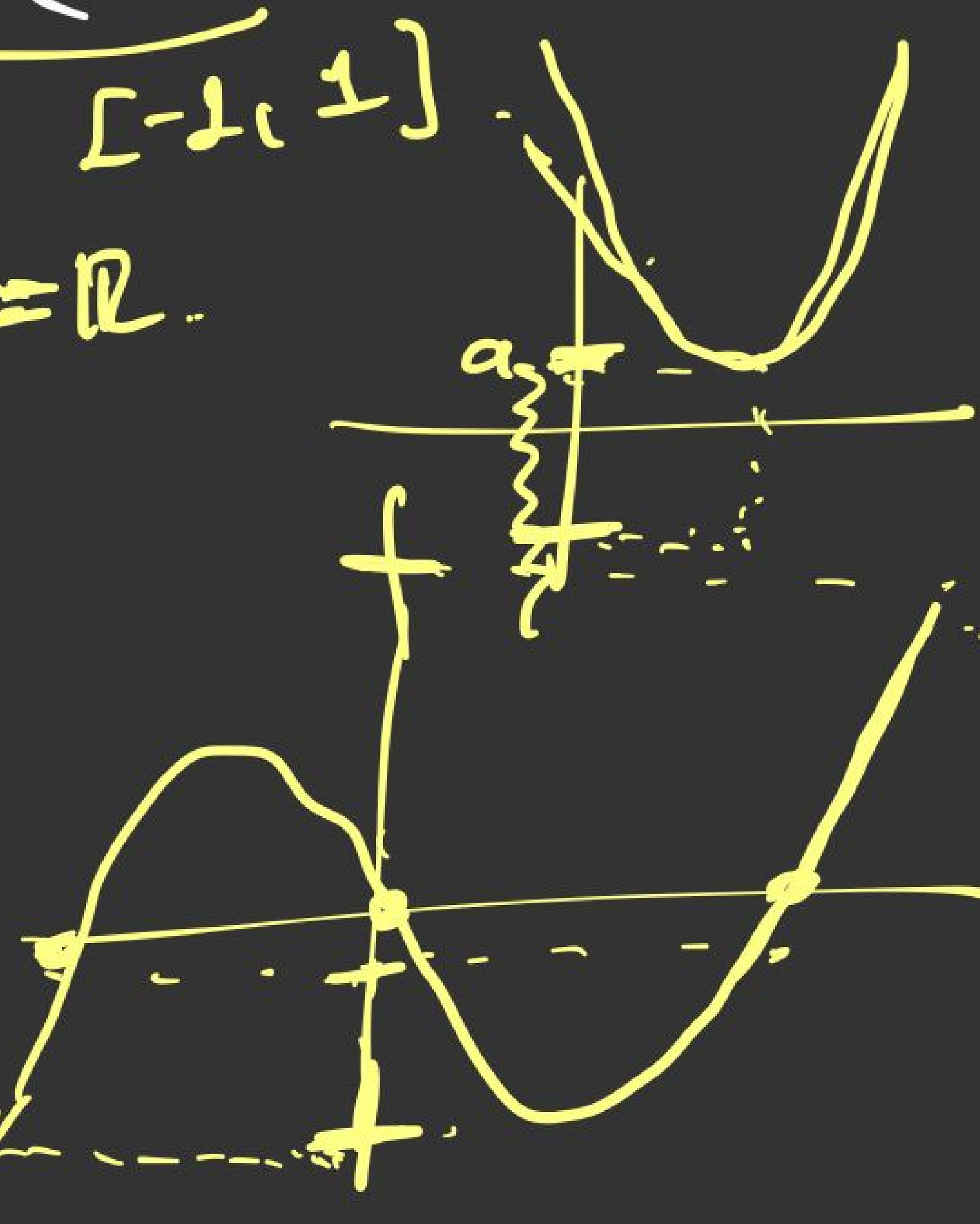
Since f is a sum of sine function and a polynomial function, and since both of them are defined everywhere on $\mathbb{R}$, f will be defined everywhere on $\mathbb{R}$. So, $\text{Dom} f = \mathbb{R} = \text{Ran} f^{-1}$.

$[a, \infty)$.

$$f(x) = x^5 + 7x - 2 - \underbrace{2\sin(\pi x)}_{[-2,\,2]}$$

Since $x^5 + 7x - 2$ has odd degree, $\text{Ran}(x^5 + 7x - 2) = \mathbb{R}$.





Therefore, f has its range as $\mathbb{R}$ ($\mathrm{Ran} f = \mathbb{R}$
$= \mathrm{Dom} f^{-1}$).

(c) Compute $\frac{df^{-1}}{dx}(6)$. $(f^{-1})'(6)$

$$f(f^{-1}(x)) = x.$$

$$\Rightarrow f'(f^{-1}(x)) \cdot \underline{(f^{-1})'(x)} = 1.$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

$$(f^{-1})'(6) = \frac{1}{f'(\underline{f^{-1}(6)})}$$

$$f(x) = x^5 + 7x - 2 - 2\sin(\pi x).$$

$$6 = x^5 + 7x - 2 - 2\sin(\pi x).$$

$$\Rightarrow 8 = x^5 + 7x - 2\sin(\pi x). \Rightarrow x = 1.$$

guess!

So $f(1) = 6 \Rightarrow f^{-1}(6) = 1$.

$$(f^{-1})'(6) = \frac{1}{f'(1)}.$$

$$f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x).$$

$$f'(1) = 5 + 7 - 2\pi.(-1)$$

$$= 12 + 2\pi.$$

So $$(f^{-1})'(6) = \frac{1}{12+2\pi}.$$

#5: Find $\frac{dy}{dx}$ at $x=1$ if the differentiable function $y=f(x)$ is defined by

$$2xe^{y}+ye^{x}=3e^{x}, \quad f(1)=1 \Rightarrow y=1.$$

$y=f(1)$

Since y cannot be written as a function of x explicitly, we have to use implicit differentiation to find $\left.\frac{dy}{dx}\right|_{x=1}$.

Take the derivative of each side wrt x, but keep in mind that y is depending on x.

$$\left.\frac{d}{dx}\left(2xe^{y}+ye^{x}\right)\right|_{\substack{x=1\\y=1}} = \left.\frac{d}{dx}\left(3e^{x}\right)\right|_{\substack{x=1\\y=1.}}$$

$$\left.\left(\overbrace{2.e^{y}+\underset{1}{2x}.e^{y}.\frac{dy}{dx}}+\overbrace{1.\frac{dy}{dx}e^{x}+y.e^{x}}\right)\right|_{\substack{x=1\\y=1}} = \left.3e^{x}\right|_{\substack{x=1\\y=1.}}$$

$$\cancel{2e}+2e.\left.\frac{dy}{dx}\right|_{x=1}+e\left.\frac{dy}{dx}\right|_{x=1}+\cancel{e}=\cancel{3e}.$$

$$3e.\left.\frac{dy}{dx}\right|_{x=1}=0 \Rightarrow \boxed{\left.\frac{dy}{dx}\right|_{x=1}=0}.$$

#6: Consider the curve given by the following implicit equation $\tan(x+y) = \sin(xy)$. Find the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$.

$$\underset{x_0}{\sqrt{\pi}}, \quad \underset{y_0}{-\sqrt{\pi}}$$

$$y = m(x - x_0) + y_0.$$

In the implicit case:

$$m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}.$$

In order to write the eqn of the tangent line, we need a pt and slope.

In explicit case: $m = f'(x_0)$.

$$m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$

$$\left.\frac{d}{dx}\left[\tan(x+y)\right]\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \left.\frac{d}{dx}\left(\sin(xy)\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})}$$

$$\left.\sec^2(x+y)\cdot\left(1 + \frac{dy}{dx}\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \left.\cos(xy)\cdot\left(y + x\cdot\frac{dy}{dx}\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})}$$

$$\underbrace{\sec^2(\underbrace{\sqrt{\pi} - \sqrt{\pi}}_{0})}_{=1}\cdot\left(1 + \left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}\right) = \underbrace{\cos(-\pi)}_{=-1}\cdot\left(-\sqrt{\pi} + \sqrt{\pi}\cdot\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}\right)$$

$$1 + \left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \sqrt{\pi} - \sqrt{\pi}\cdot\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}$$

$$(1+\sqrt{\pi})\cdot \left.\frac{dy}{dx}\right|_{(\sqrt{\pi},-\sqrt{\pi})} = \sqrt{\pi}-1.$$

$$\boxed{\left.\frac{dy}{dx}\right|_{(\sqrt{\pi},-\sqrt{\pi})} = \frac{\sqrt{\pi}-1}{1+\sqrt{\pi}}} = m.$$

$$\boxed{y = \left(\frac{\sqrt{\pi}-1}{1+\sqrt{\pi}}\right)\cdot(x-\sqrt{\pi})-\sqrt{\pi}}$$