REC · V

#1: Show that $x^3+x^2+3x+7=0$ has exactly one real root.

Recall: Intermediate Value Thm

If f is cont. on $[a,b]$ and if s is a number between $f(a)$ & $f(b)$ then there exists some $c \in [a,b]$ s.t $f(c)=s$.

Mean Value Thm

If f is cont on $[a,b]$ and differentiable on (a,b) then there exists some $c \in (a,b)$ s.t $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Existence: Let $f(x) = x^3+x^2+3x+7$. Then observe that f is cont. everywhere.

$f(0) = 7 > 0$.

$f(-2) = \underbrace{-8+4}_{-4} \underbrace{-6+7}_{1} = -3 < 0$.

Since f is cont. on $[-2,0]$ s.t. $\underbrace{f(-2)}_{=-3} < 0 < \underbrace{f(0)}_{=7}$ by IVT, there exists some $c \in [-2,0]$ s.t $f(c)=0$.

So, f has at least one root.

Uniqueness: Assume f has two different roots, say c & d. $(c<d)$. Then $f(c)=0=f(d)$.

Since f is cont. on $[c,d]$ and f is diff'ble on (c,d) by MVT, there exists some $e\in(c,d)$ s.t. $f'(e)=\frac{f(d)-f(c)}{d-c}=0$.

$f'(x)=3x^2+2x+3$. (ax^2+bx+c). $\lightning$ (Contradiction)

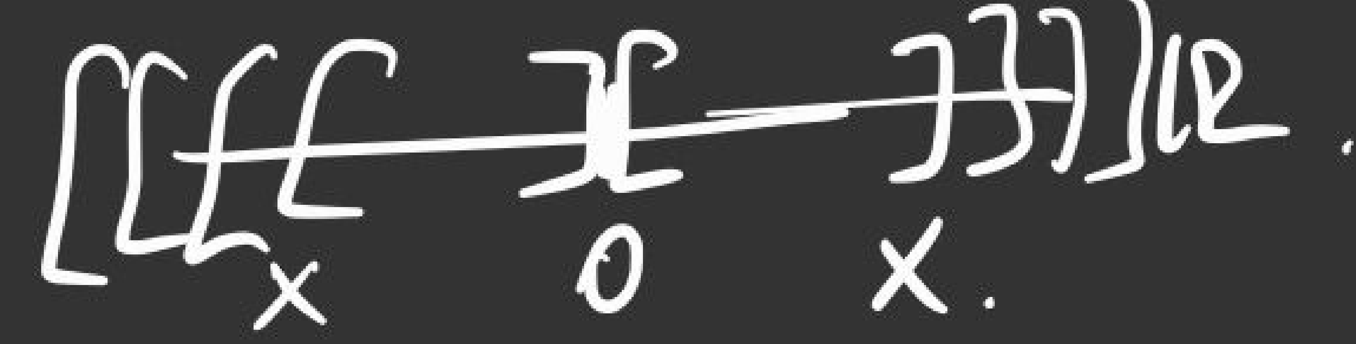
$\Delta=b^2-4ac=4-4.3.3=-32<0$.

Therefore, there is no such e $(f'(e)=0)$.

Hence, f cannot have two different roots.

So, f has exactly one root.

#2: Prove that $e^x \geq x+1$ for all $x \in \mathbb{R}$.



Case 1: $x=0$.

$\underbrace{e^0}_{1} \geq \underbrace{0+1}_{1}$ ✓ Inequality holds.

Let $f(t)=e^t$. Observe that f is continuous and diff'ble everywhere. $f'(t)=e^t$.

Case 2: $x<0$.

(i) f is cont. on $[x,0]$

(ii) f is diff'ble on $(x,0)$

By MVT, there is some $c \in \boxed{(x,0)}$ s.t

$$\frac{1-e^x}{-x} = \frac{e^0-e^x}{-x} = \frac{f(0)-f(x)}{0-x} = f'(c) = e^c < 1 \quad (c<0)$$

$$\frac{1-e^x}{-x} < 1 \Rightarrow 1-e^x < -x \Rightarrow \boxed{1+x < e^x}$$

Case 3: $x>0$.

(i) f is cont. on $[0,x]$

(ii) f is diff'ble on $(0,x)$.

By MVT, there exists some $d \in (\underset{a}{0}, \underset{b}{x})$ s.t

$$\frac{e^x - 1}{x} = \frac{e^x - e^0}{x} = \frac{f(x) - f(0)}{x - 0} = f'(d) = e^d \underset{(d>0)}{>} 1$$

$$\frac{e^x - 1}{x} > 1 \Rightarrow e^x - 1 > x.$$

$$\Rightarrow \boxed{e^x > x + 1.}$$

Therefore, $e^x \geq x + 1 \quad \forall x \in \mathbb{R}.$

#3: Let f be a function defined on $[0,9]$ and f', f'' exist on $(0,9)$. If $f(1)=-2$, $f(3)=5$, $f(4)=6$ and $f(8)=20$, then show that there is a number $c \in (0,9)$ such that $f''(c)=0$.

[Thm: If f is differentiable at a then it must be continuous at a. (The converse is not true!)].

Since f' exists on $(0,9)$, it means that f is differentiable on $(0,9)$. By thm, f must be cont. on $(0,9)$.

(i) f is cont. on $[1,3]$
(ii) f is diff'ble on $(1,3)$.

By MVT, there is some $d \in (1,3)$ s.t.

$$f'(d) = \frac{f(3)-f(1)}{3-1} = \frac{5-(-2)}{2} = \frac{7}{2}.$$

Now, (i) f is cont on $[4,8]$
(ii) f is diff'ble on $(4,8)$.

By MVT, there is some $e \in (4,8)$ s.t

$$\underline{f'(e)} = \frac{f(8)-f(4)}{8-4} = \frac{20-6}{4} = \frac{14}{4} = \underline{\underline{\frac{7}{2}}}.$$

Observe that since f'' exists on $(0,9)$, it means f' is diff'ble on $(0,9)$. Therefore by Thm, f' is cont on $(0,9)$.

Now, (i) f' is cont. on $[d,e]$.

(ii) f' is diff'ble on (d,e).

By MVT, there is some $c \in \underline{\underline{(d,e)}} \subset (0,9)$.

s.t $$f''(c) = \frac{f'(e)-f'(d)}{e-d} = \frac{7/2 - 7/2}{e-d} = 0.$$

#4: Given $f(x) = x^5 + 7x - 2\sin(\pi x) - 2$.

(a) Show that f^{-1} exists.

[$f : \text{Dom} f \to \text{Ran} f$

Facts: ① If f is 1-1, then f^{-1} exists.

② Every strictly increasing function is 1-1.

③ f is strictly increasing iff $f' > 0$.]

$$f'(x) = \underbrace{5x^4}_{\geq 0} + \underbrace{7 - 2\pi\cos(\pi x)}_{> 0} > 0$$

since

$$-1 \leq \cos \pi x \leq 1.$$

$$-2\pi \leq -2\pi\cos(\pi x) \leq 2\pi$$

$$0 < 7 - 2\pi \leq 7 - 2\pi\cos(\pi x) \leq 7 + 2\pi.$$

Therefore, since $f'(x) > 0$ $\forall x \in \text{Dom} f$, f is strictly increasing. So, by Fact 2, f is 1-1.

Therefore, f^{-1} exists.

(b) Find the domain and range of $\underline{f^{-1}}(x)$.

$f^{-1}: \underbrace{\text{Ran} f}_{=\text{Dom} f^{-1}} \to \underbrace{\text{Dom} f}_{=\text{Ran} f^{-1}}$. $\qquad f(x) = x^5 + 7x - 2 - 2\sin(\pi x)$.

Since $x^5 + 7x - 2$ is defined everywhere on $\mathbb{R}$ and since $-2\sin(\pi x)$ is also defined everywhere on $\mathbb{R}$, $f(x) = x^5 + 7x - 2 - 2\sin(\pi x)$ will also be defined everywhere. Therefore, we have

$\text{Dom} f = \mathbb{R}\ (= \text{Ran} f^{-1})$.

<u>Fact</u>: If f is an odd degree polynomial, it will have range $\mathbb{R}$. If it is an even degree polynomial, then its range is either bdd from below or above.

Since $\deg(x^5 + 7x - 2) = 5$ (odd), its range is $\mathbb{R}$. $f(x) = x^5 + 7x - 2 - \underbrace{2\sin(\pi x)}_{-1 \leq \ \leq 1}$.

So, $\text{Ran} f = \mathbb{R} = \text{Dom} f^{-1}$.

(c) Compute $\frac{df^{-1}}{dx}(6)$

$f(x) = x^5 + 7x - 2 - 2\sin(\pi x)$.

$\rightarrow \quad f(f^{-1}(x)) = x$.

$1 = f'(f^{-1}(x)) \cdot (f^{-1})'(x)$.

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(1)}$$

$6 = x^5 + 7x - 2 - 2\sin(\pi x)$

$8 = x^5 + 7x - 2\sin(\pi x) \Rightarrow x = 1 \rightarrow f(1) = 6.$

guess!

$\Rightarrow f^{-1}(6) = 1.$

$f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x)$

$f'(1) = 12 + 2\pi$ Hence, $(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{12+2\pi}$

#5: Find $\frac{dy}{dx}$ at $x=1$ if the differentiable function $y=f(x)$ is defined by

$2xe^y + ye^x = 3e^x$, $f(1)=1$ $\Rightarrow y=1$. $y=f(x)$ ($x=1$)

$$\frac{d}{dx}\left(\underline{2xe^y}+\underline{ye^x}\right)\Big|_{\substack{x=1\\y=1}} = \frac{d}{dx}\left(3e^x\right)\Big|_{\substack{x=1\\y=1}}$$

$$\left(\overbrace{2.e^y + \underset{1}{2x}.e^{y}.\frac{dy}{dx}}+\overbrace{1.\frac{dy}{dx}e^x+y.e^x}\right)\Big|_{\substack{x=1\\y=1}} = 3e^x\Big|_{\substack{x=1\\y=1}}$$

$$\cancel{2e} + 2e.\frac{dy}{dx}\Big|_{x=1} + e\frac{dy}{dx}\Big|_{x=1} + \cancel{e} = \cancel{3e}.$$

$$3e.\frac{dy}{dx}\Big|_{x=1} = 0 \Rightarrow \boxed{\frac{dy}{dx}\Big|_{x=1} = 0}.$$

#6: Consider the curve given by the following implicit equation $\tan(x+y) = \sin(xy)$. Find the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$.

$$\underset{x_0}{\sqrt{\pi}}, \quad \underset{y_0}{-\sqrt{\pi}}$$

$$y = m(x - x_0) + y_0.$$

$$m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$

$$\left.\frac{d}{dx}\left[\tan(x+y)\right]\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \left.\frac{d}{dx}\left(\sin(xy)\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})}$$

$$\left.\sec^2(x+y)\cdot\left(1 + \frac{dy}{dx}\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \left.\cos(xy)\cdot\left(y + x\cdot\frac{dy}{dx}\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})}$$

$$\underbrace{\sec^2(\underbrace{\sqrt{\pi} - \sqrt{\pi}}_{0})}_{=1}\cdot\left(1 + \left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}\right) = \underbrace{\cos(-\pi)}_{=-1}\cdot\left(-\sqrt{\pi} + \sqrt{\pi}\cdot\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}\right)$$

$$\underline{1} + \underline{\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}} = \sqrt{\pi} - \sqrt{\pi}\cdot\underline{\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}}$$

$$(1+\sqrt{\pi})\cdot \left.\frac{dy}{dx}\right|_{(\sqrt{\pi},-\sqrt{\pi})} = \sqrt{\pi}-1.$$

$$\boxed{\left.\frac{dy}{dx}\right|_{(\sqrt{\pi},-\sqrt{\pi})} = \frac{\sqrt{\pi}-1}{1+\sqrt{\pi}}} = m.$$

$$\boxed{y = \left(\frac{\sqrt{\pi}-1}{1+\sqrt{\pi}}\right)\cdot(x-\sqrt{\pi})-\sqrt{\pi}}$$