REC · V

#1: Show that $x^3+x^2+3x+7=0$ has exactly one real root.

[Recall: Intermediate Value Thm

If f is cont. on $[a,b]$ and if s is a number between $f(a)$ & $f(b)$ then there exists some $c\in[a,b]$ s.t $f(c)=s$.

Mean Value Thm

If f is cont on $[a,b]$ and differentiable on (a,b) then there exists some $c\in(a,b)$ s.t $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.]

Existence: Let $f(x)=x^3+x^2+3x+7$. Then observe that f is continuous & diff'ble everywhere.

$$f(0)=7>0$$

$$f(-2)=\underbrace{-8+4}_{-4}-\underbrace{6+7}_{+1}=-3<0.$$

Since f is cont on $[-2,0]$ & $\underbrace{f(-2)}_{=-3}<0<\underbrace{f(0)}_{=7}$ then by IVT, there exists some $c\in[-2,0]$ s.t $f(c)=0$.

Therefore $f(x)=x^3+x^2+3x+7$ has at least one root.

Uniqueness : Assume that there are two different roots, say c & d. $(c<d)$ Then $\underline{f(c)=0}=\underline{f(d)}$.

Since f is cont. on $[c,d]$ and f is differentiable on (c,d), then by MVT, there exists some $e\in(c,d)$ s.t

$$f'(e)=\frac{f(d)-f(c)}{d-c}=\frac{0-0}{d-c}=0.$$

$$f(x)=x^3+x^2+3x+7 \Rightarrow f'(x)=3x^2+2x+3.$$

$(ax^2+bx+c \qquad \Delta=b^2-4ac=4-4.3.3=-32<0$ ↯

So, $f'(x)$ cannot be zero for any real x.

Therefore, there is no such e.

Hence f has a unique root.

\# 2: Prove that $e^x \geq x+1$ for all $x \in \mathbb{R}$.

Case 1: $x = 0$.

$$\underbrace{e^0}_{1} \geq \underbrace{0+1}_{1}$$ ✓ Inequality holds.

Let $f(t) = e^t$. Observe that f is cont & diff'ble everywhere. $f'(t) = e^t$.

Case 2: $x < 0$

(i) f is cont on $[x, 0]$

(ii) f is diff'ble on $(x, 0)$.

By MVT, there is some $c \in (x, 0)$ s.t

$$\frac{1-e^x}{-x} = \frac{e^0 - e^x}{-x} = \frac{f(0)-f(x)}{0-x} = f'(c) = e^c < 1.$$

$c < 0$

$$\Rightarrow \frac{1-e^x}{-x} < 1 \Rightarrow 1 - e^x < -x \Rightarrow 1 + x < e^x.$$

Case 3: $x > 0$.

(i) f is cont on $[0, x]$

(ii) f is diff'ble on $(0, x)$.

By mvT, there exists $d \in (0, x)$ s.t

$$\frac{e^x - 1}{x} = \frac{e^x - e^0}{x} = \frac{f(x) - f(0)}{x - 0} = f'(d) = e^d > 1.$$

$d > 0$

$$\frac{e^x - 1}{x} > 1 \Rightarrow e^x - 1 > x \Rightarrow e^x > 1 + x.$$

Hence $e^x > 1 + x$ for $x \in \mathbb{R} \setminus \{0\}$ and for $x = 0$ $e^x = 1 + x$. Therefore,

$$e^x \geq 1 + x \quad \forall x \in \mathbb{R}.$$

#3: Let f be a function defined on $[0,9]$ and f', f'' exist on $(0,9)$. If $f(1)=-2$, $f(3)=5$, $f(4)=6$ and $f(8)=20$, then show that there is a number $c \in (0,9)$ such that $f''(c)=0$.

[Thm: If f is differentiable at a then it must be continuous at a.]

We can say that, since f' exists on $(0,9)$, f is cont on $(0,9)$ by Thm.

(i) f is cont on $[1,3]$

(ii) f is diff'ble on $(1,3)$ $(\subset (0,9))$.

By MVT, $\exists\, d \in (1,3)$ s.t

$$f'(d) = \frac{f(3)-f(1)}{3-1} = \frac{5-(-2)}{2} = \frac{7}{2}.$$

Now, (i) f is cont on $[4,8]$ $(\subset (0,9))$

(ii) f is diff'ble on $(4,8)$ $(\subset (0,9))$.

By MVT, $\exists\, e \in (4,8)$ s.t

$$f'(e) = \frac{f(8)-f(4)}{8-4} = \frac{20-6}{4} = \frac{14}{4} = \frac{7}{2}.$$

Since f'' exists on $(0,9)$, it means that f' is diff'ble on $(0,9)$. So by thm, f' is continuous on $(0,9)$.

Therefore, (i) f' is cont on $[d,e]$ $(\subset (0,9))$

(ii) f' is diff'ble on (d,e) $(\subset (0,9))$.

By MVT, there exists $c \in (d,e) \subset (0,9)$.

s.t $f''(c) = \dfrac{f'(e) - f'(d)}{e-d} = \dfrac{7/2 - 7/2}{e-d} = 0.$

#4: Given $f(x) = x^5 + 7x - 2\sin(\pi x) - 2$.

(a) Show that f^{-1} exists.

[$f : \text{Dom} f \to \text{Ran} f$

Facts: ① If f is 1-1, then f^{-1} exists.

② Every strictly increasing function is 1-1.

③ f is strictly increasing if $f' > 0$.]

$$f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x).$$

we know that $-1 \leq \cos(\pi x) \leq 1$.

$$\Rightarrow \quad -2\pi \leq -2\pi\cos(\pi x) \leq 2\pi.$$

$$\Rightarrow \quad 0 < 7 - 2\pi \leq 7 - 2\pi\cos(\pi x) \leq 7 + 2\pi.$$

Therefore $f'(x) = \underbrace{5x^4}_{\geq 0} + \underbrace{7 - 2\pi\cos(\pi x)}_{> 0} > 0$

Hence, by Facts above, f^{-1} exists.

(b) Find the domain and range of $f^{-1}(x)$.

$f : \text{Dom} f \to \text{Ran} f$ $\qquad$ $f^{-1} : \text{Ran} f \to \text{Dom} f$.

$f(x) = x^5 + 7x - 2 - 2\sin(\pi x)$. $\qquad$ $= \text{Dom} f^{-1}$ $\qquad$ $= \text{Ran} f^{-1}$.

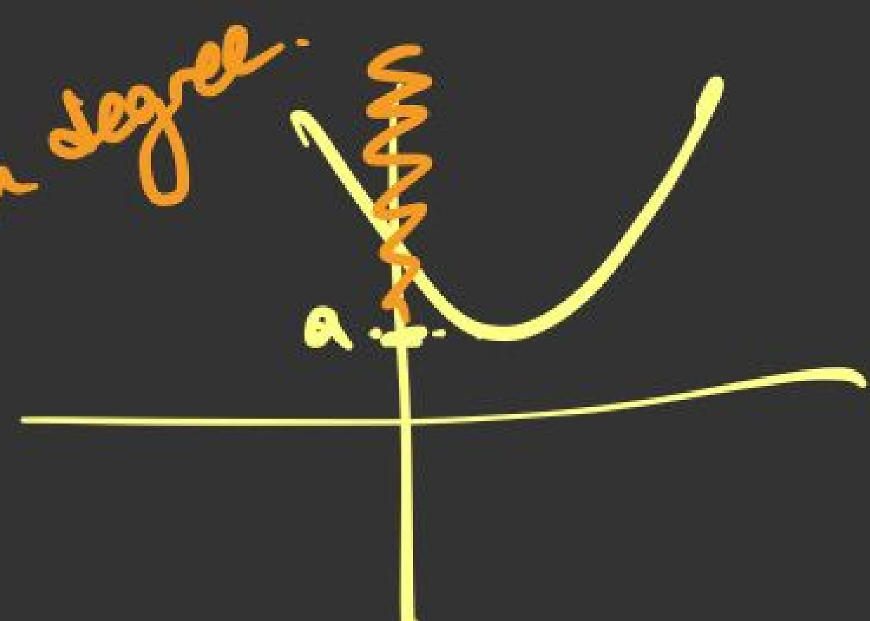
Since $x^5 + 7x - 2$ and $\sin(\pi x)$ are defined everywhere on $\mathbb{R}$, $f(x) = x^5 + 7x - 2 + \sin(\pi x)$ must also be defined everywhere on $\mathbb{R}$ so we will have

$\text{Dom} f = \mathbb{R} = \text{Ran} f^{-1}$.

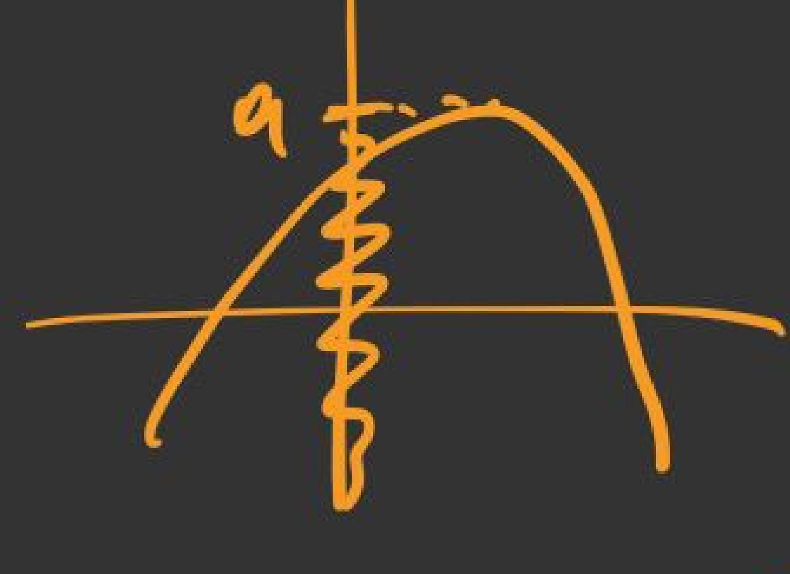
$\text{Ran}(x^5 + 7x - 2) = \mathbb{R}$ since $\deg(x^5 + 7x - 2) = 5$ which is odd.

[Fact: An odd degree polynomial has its range as whole real numbers.
An even degree polynomial has its range as an interval either bounded from below or above.].

even degree.

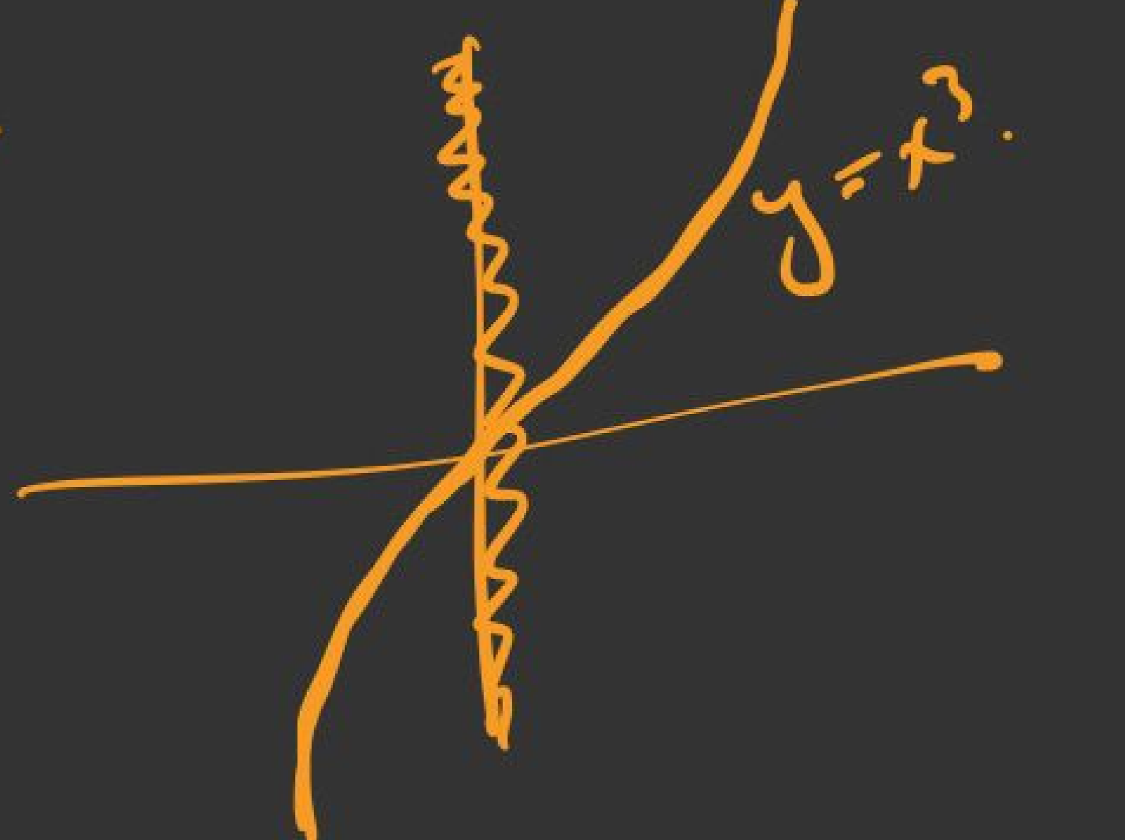


$[a, \infty)$



$(-\infty, a]$.

odd degree.



$\mathbb{R} = \text{ran}(x^3)$

(c) Compute $\frac{df^{-1}}{dx}(6)$

$f(x) = x^5 + 7x - 2 - 2\sin(\pi x).$

- $f(f^{-1}(x)) = x.$

$1 = f'(f^{-1}(x)) \cdot (f^{-1})'(x).$

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.}$$

$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(1)}$

$6 = x^5 + 7x - 2 - 2\sin(\pi x)$

$\Rightarrow 8 = \underbrace{x^5 + 7x}_{=8} - \underbrace{2\sin(\pi x)}_{=0}.$ $\Rightarrow x = 1.$

$\Rightarrow f(1) = 6.$

$f^{-1}(6) = 1.$

$f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x).$

$f'(1) = 12 + 2\pi$

$$\Rightarrow \boxed{(f^{-1})'(6) = \frac{1}{12+2\pi}}$$

#5: Find $\frac{dy}{dx}$ at $x=1$ if the differentiable function

$y=f(x)$ is defined by

$2xe^{y}+ye^{x}=3e^{x}$, $f(1)=1$. $\quad y=f(x) \Rightarrow y=1$.

$$\frac{d}{dx}\left[2x.e^{y}+y.e^{x}\right]\Big|_{\substack{x=1\\y=1}}=\frac{d}{dx}\left(3e^{x}\right)\Big|_{\substack{x=1\\y=1}}$$

$$\underbrace{2e^{y}+2x.e^{y}.\frac{dy}{dx}}+\underbrace{1.\frac{dy}{dx}.e^{x}+y.e^{x}}\Big|_{\substack{x=1\\y=1}}=3.e^{x}\Big|_{\substack{x=1\\y=1}}$$

$$\cancel{2.e^{1}}+2e^{1}\frac{dy}{dx}\Big|_{x=1}+e^{1}\frac{dy}{dx}\Big|_{x=1}+\cancel{e^{1}}=\cancel{3e^{1}}$$

$$\frac{dy}{dx}\Big|_{x=1}(3e)=0 \quad\Rightarrow\quad \frac{dy}{dx}\Big|_{x=1}=0.$$

#6: Consider the curve given by the following implicit equation $\tan(x+y) = \sin(xy)$. Find the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$.

$$\underset{x_0}{\sqrt{\pi}}, \underset{y_0}{-\sqrt{\pi}}$$

$$y = m \cdot (x - x_0) + y_0.$$

$$m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$

$$\left.\frac{d}{dx}\left(\tan(x+y)\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \left.\frac{d}{dx}\left(\sin(xy)\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})}.$$

$$\left.\sec^2(x+y) \cdot \left(1 + \frac{dy}{dx}\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})} = \left.\cos(xy)\left(y + x \cdot \frac{dy}{dx}\right)\right|_{(\sqrt{\pi}, -\sqrt{\pi})}$$

$\Rightarrow$

$$\underbrace{\sec^2(\underbrace{\sqrt{\pi} - \sqrt{\pi}}_{=0})}_{=1.}\left(1 + \left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}\right) = \underbrace{\cos(-\pi)}_{=-1} \cdot \left(-\sqrt{\pi} + \sqrt{\pi} \cdot \left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}\right).$$

$$\underline{1} + \underline{\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}} = \sqrt{\pi} - \sqrt{\pi} \cdot \underline{\left.\frac{dy}{dx}\right|_{(\sqrt{\pi}, -\sqrt{\pi})}}.$$

$$\Rightarrow \quad (1+\sqrt{\pi}) \cdot \left.\frac{dy}{dx}\right|_{(\sqrt{\pi},-\sqrt{\pi})} = \sqrt{\pi} - 1.$$

$$\left.\frac{dy}{dx}\right|_{(\sqrt{\pi},-\sqrt{\pi})} = \frac{\sqrt{\pi}-1}{1+\sqrt{\pi}} = m.$$

$$\boxed{y = \left(\frac{\sqrt{\pi}-1}{1+\sqrt{\pi}}\right)(x-\sqrt{\pi}) - \sqrt{\pi}}$$