REC - IV

#1: Consider the curve given by $y = \frac{16}{x} - x^2 = f(x)$. Find the points where this curve has a horizontal tangent line.

m = the slope of the tangent line at the point $(a, f(a))$

$$\boxed{m = f'(a)}$$

$m = 0$ we are looking for the pt where $f'(a) = 0$.

$$y = f(x) = 16 \cdot x^{-1} - x^2$$

$$f'(x) = -16 \cdot x^{-2} - 2x$$

$$f'(x) = 0 \iff -16x^{-2} - 2x = 0. \quad -\frac{16}{x^2} - \frac{2x}{1}\left(\frac{x^2}{x^2}\right) = 0.$$

$$\frac{-16 - 2x^3}{x^2} = 0. \iff -16 - 2x^3 = 0 \iff x^3 = -8$$

$$\Rightarrow x = -2. \quad \Rightarrow a = -2. \quad f(a) = \frac{16}{-2} - 4 = -12.$$

At the point $(-2, -12)$, $y = f(x)$ will have a horizontal tangent line.

#2: Calculate the derivative of the given function using the definition of the derivative.

$$y = \frac{1}{\sqrt{1+x^2}} = f(x).$$

[Recall: The derivative of a function f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$].

$$f'(x) = \lim_{h \to 0} \frac{\dfrac{1}{\sqrt{1+(x+h)^2}} - \dfrac{1}{\sqrt{1+x^2}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{\sqrt{1+(x+h)^2} \cdot \sqrt{1+x^2} \cdot h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{\sqrt{1+(x+h)^2} \cdot \sqrt{1+x^2} \cdot h} \cdot \frac{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}$$

$$= \lim_{h\to 0} \frac{\cancel{(1+x^2)} - (\cancel{1+x^2}+2xh+h^2)}{h\cdot\sqrt{1+x^2}\cdot\sqrt{1+(x+h)^2}\,\left(\sqrt{1+x^2}+\sqrt{1+(x+h)^2}\right)}$$

$$= \lim_{h\to 0} \frac{-2xh-h^2 \to \cancel{h}(-2x-h)}{\cancel{h}\cdot\sqrt{1+x^2}\cdot\sqrt{1+(x+h)^2}\,\left(\sqrt{1+x^2}+\sqrt{1+(x+h)^2}\right)}$$

$$= \frac{-2x}{2\cdot\left(\sqrt{1+x^2}\right)^3} = \boxed{\frac{-x}{(1+x^2)^{3/2}}} = f'(x).$$

#3: How should the function $f(x) = x^2 \cdot \sin(1/x)$ be defined at $x=0$ so that it is continuous at $x=0$? Is it then differentiable?

[f is cont at $x=a \iff \lim_{x\to a} f(x) = f(a)$.]

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \cdot \sin(1/x).$$

Let's use Squeeze Theorem.

$-1 \le \sin x \le 1 \quad \forall x \in \mathbb{R}.$ $-c \;\; 0 \;\; c.$

$-1 \le \sin x \le 1 \quad \forall x \in (-c, c)$

$-1 \le \sin(1/x) \le 1 \quad \forall x \in (-c, c).$

$(x^2 \geq 0).$

$-x^2 \le x^2 \cdot \sin(1/x) \le x^2$

$$\underbrace{\lim_{x \to 0} -x^2}_{=0} \le \lim_{x \to 0} x^2 \sin(1/x) \le \underbrace{\lim_{x \to 0} x^2}_{=0}.$$

By Squeeze Theorem, $\lim_{x \to 0} x^2 \cdot \sin(1/x) = 0.$

Therefore,

Define $f(x) = \begin{cases} x^2 \cdot \sin(1/x) & \text{if } x \neq 0. \\ 0. & \text{if } x = 0. \end{cases}$

Now, f becomes continuous at $x = 0.$

[Recall: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$]

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \cdot \sin(1/h) - 0}{h}$$

$$= \lim_{h\to 0} h \cdot \sin(1/h) = 0$$

by Squeeze Thm

$-1 \le \sin(1/h) \le 1. \quad \forall h \in (-c, c)$

$-h \le h \cdot \sin(1/h) \le h.$

$\lim_{h\to 0}(-h) = 0 \qquad \lim_{h\to 0} h = 0$

$\therefore$ by Squeeze Thm

$\lim_{h\to 0} h \cdot \sin(1/h) = 0.$

WARNING: There is a mistake in the above step. when h approaches to zero, it is not always positive. Therefore, there will be two different inequalities, and left limit and right limit will be computed seperately.

For $h > 0$

$-1 \le \sin(1/h) \le 1 \quad \forall h \in \mathbb{R}$

$\Downarrow \; h > 0$

$-h \le h \cdot \sin(1/h) \le h \quad \forall h \in (0, c)$

$\downarrow$ Take lim as $h \to 0^+$.

$\underbrace{\lim_{h\to 0^+} -h}_{=0} \le \lim_{h\to 0^+} h \cdot \sin(1/h) \le \underbrace{\lim_{h\to 0^+} h}_{=0}.$

$\therefore \lim_{h\to 0^+} h\sin(1/h) = 0$ by Squeeze Thm.

For $h < 0$

$-1 \le \sin(1/h) \le 1 \quad \forall h \in \mathbb{R}$

direction will change! $\Downarrow \; h < 0$

$h \le h\sin(1/h) \le -h. \quad \forall h \in (-c, 0)$

$\downarrow$ Take lim as $h \to 0^-$

$\underbrace{\lim_{h\to 0^-} h}_{=0} \le \lim_{h\to 0^-} h \cdot \sin(1/h) \le \underbrace{\lim_{h\to 0^-} -h}_{=0}.$

$\therefore \lim_{h\to 0^-} h \cdot \sin(1/h) = 0.$ by Squeeze Thm.

Hence, $\lim_{h \to 0} h \cdot \sin(1/h) = 0$. by Squeeze Thm.

So, when you are multiplying each side of the inequality with something, make sure that it is either always positive or negative. If it is always positive, then keep the direction of the inequality. If it is always negative, then reverse the direction of the inequality. Or if it is sometimes negative sometimes positive, just like in this case, consider two seperate cases to apply squeeze thm and take left limit, right limit seperately.

L

#4: Let $g(x)$ be continuous at $x=a$ and consider the function $f(x) = (x-a) \cdot g(x)$.

Find $f'(a)$ in terms of g.

[Product Rule: $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$]

$$f'(x) = ((x-a) \cdot g(x))' \underset{\text{Product Rule}}{=} \underbrace{(x-a)'}_{=1} \cdot g(x) + (x-a) \cdot g'(x).$$

$$\Rightarrow f'(x) = g(x) + (x-a) \cdot g'(x).$$

$$f'(a) = g(a) + \underbrace{(a-a)}_{=0} \cdot g'(a) = \underline{\underline{g(a)}}.$$

#5: Given that $f(1) = 2$, $f'(1) = 1$, $g(1) = 3$ and $g'(1) = 4$. Calculate the followings:

(a) $\left.\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)+x}\right)\right|_{x=1}$.

Quotient Rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$.

$$\left.\left(\frac{f(x)}{g(x)+x}\right)'\right|_{x=1} = \left.\frac{f'(x)(g(x)+x) - f(x) \cdot (g(x)+x)'}{(g(x)+x)^2}\right|_{x=1}.$$

$$= \left.\frac{f'(x) \cdot (g(x)+x) - f(x) \cdot (g'(x)+1)}{(g(x)+x)^2}\right|_{x=1}.$$

$$= \frac{\overbrace{f'(1)}^{1}\cdot(\overbrace{g(1)}^{3}+1) - \overbrace{f(1)}^{2}\cdot(\overbrace{g'(1)}^{4}+1)}{(\underbrace{g(1)}_{3}+1)^2}$$

$$= \frac{4-10}{16} = -\frac{3}{8}.$$

(b) $\frac{d}{dx}\left(f^2(x)\cdot g(x)\right)\Big|_{x=1}$.

$$\left(f^2(x)\cdot g(x)\right)'\Big|_{x=1} = \left(f^2(x)\right)'\cdot g(x) + f^2(x)\cdot g'(x)\Big|_{x=1} =$$

[Chain Rule $\left(f(g(x))\right)' = f'(g(x))\cdot g'(x)$].

[$f^2(x) = \underbrace{x^2}_{2x} \circ f(x)$

$= 2f(x)\cdot f'(x)$.

OR $f^2(x) = f(x)\cdot f(x) = f'(x)\cdot f(x) + f(x)\cdot f'(x)$

From product rule.

$= 2\cdot f'(x)\cdot f(x)$].

$$= 2f(x)\cdot f'(x)\cdot g(x) + f^2(x)\cdot g'(x) \Big|_{x=1}$$

$$= 2\underbrace{f(1)}_{2}\cdot\underbrace{f'(1)}_{1}\cdot\underbrace{g(1)}_{3} + \underbrace{f^2(1)}_{2^2}\cdot\underbrace{g'(1)}_{4}$$

$$= 12 + 16 = 28$$

Exercise: (c) $\dfrac{d}{dx}\left(x^3 \cdot f(x)\right)\Big|_{x=1}$

#6: Find the derivative of the following functions

(a) $f(x) = \sqrt{3x + \sqrt{2 + \sqrt{1-x}}}$

[Chain Rule $\left(f(g(x))\right)' = f'(g(x))\cdot g'(x)$]

$$\boxed{\left(\sqrt{f(x)}\right)' = \frac{1}{2\sqrt{f(x)}}\cdot f'(x)}$$

$$f'(x) = \frac{1}{2\underbrace{\sqrt{3x+\sqrt{2+\sqrt{1-x}}}}_{=f(x)}}\cdot\left(3x+\sqrt{2+\sqrt{1-x}}\right)'$$

$$= \frac{1}{2f(x)} \cdot \left(3 + \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \underline{(2+\sqrt{1-x})'}\right)$$

$$= \frac{1}{2f(x)} \cdot \left(3 + \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \frac{1}{2\sqrt{1-x}} \underbrace{(1-x)'}_{=-1}\right)$$

$$= \frac{1}{2f(x)} \cdot \left(3 - \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \frac{1}{2\sqrt{1-x}}\right)$$

(b) $g(x) = \left(\frac{1+\sin 3x}{3-2x}\right)^{-1}$.

$$g(x) = \left(\frac{3-2x}{1+\sin 3x}\right)$$

$$g'(x) = \frac{(3-2x)' \cdot (1+\sin 3x) - (3-2x) \cdot (1+\sin 3x)'}{(1+\sin 3x)^2}$$

$$= \frac{-2 \cdot (1+\sin 3x) - (3-2x) \cdot \cos 3x \cdot 3}{(1+\sin 3x)^2}$$

(c) $h(x) = \tan\left(\frac{\pi}{\sqrt{25-x^2}}\right)$ · Exercise.

#7: (a) Suppose f is a differentiable function and $y = x/4 - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point $x = 8$. If $g(x) = \left(f(x^3)\right)^2$, find an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 2$.

$f'(x_0) = m$.

$(x_0, f(x_0))$

$x_0 = 8$

$$y = f'(x_0)(x - x_0) + f(x_0).$$

$$y = \frac{1}{4} \cdot x - 3.$$

$\Rightarrow \boxed{f'(8) = \frac{1}{4}}$

$\boxed{f(8) = -1}$

$$y = \frac{1}{4}(x - 8) + f(8)$$

$$y = g'(2) \cdot (x-2) + g(2).$$

$$g(2) = \left(f(2^3)\right)^2 = \left(f(8)\right)^2 = (-1)^2 = 1.$$

$$g'(x) = \left[\left(f(x^3) \right)^2 \right]'$$

$$= 2 \cdot f(x^3) \cdot \left[f(x^3) \right]'$$

$$= \underline{2} \cdot f(x^3) \cdot f'(x^3) \cdot \underline{3x^2}.$$

$$g'(2) = 2 \cdot f(8) \cdot f'(8) \cdot 12$$

$$= 2 \cdot (-1) \cdot \frac{1}{4} \cdot 12 = -6.$$

$$\boxed{y = -6 \cdot (x-2) + 1.}$$

(b) If $\underline{g''(2) = 0}$ find $f''(8)$.

$$g'(x) = \underbrace{6x^2 \cdot f(x^3)} \cdot f'(x^3).$$

$$g''(x) = \left(6x^2 \cdot f(x^3) \right)' \cdot f'(x^3) + \left(6x^2 \cdot f(x^3) \right) \cdot \underline{\left(f'(x^3) \right)'}$$

$$= \underline{\left(6x^2 \cdot f(x^3) \right)'} \cdot f'(x^3) + \underline{\left(6x^2 \cdot f(x^3) \right) \cdot f''(x^3) \cdot 3x^2}.$$

$$\underset{g}{=} \left[12x \cdot f(x^3) + 6x^2 \cdot f'(x^3) \cdot 3x^2\right] \cdot f'(x^3) +$$

$$(6x^2 \cdot f(x^3)) \cdot f''(x^3) \cdot 3x^2 \cdot \quad 1/4 \qquad 1/4$$

$$0 = g''(2) = (24 \cdot \overbrace{f(8)}^{-1} + 24 \cdot \overbrace{f'(8)}^{1/4} \cdot 12) \cdot \overbrace{f'(8)}^{1/4}$$

$$+ \quad 24 \cdot \underbrace{f(8)}_{-1} \cdot f''(8) \cdot 12 \cdot$$

$$0 = \underbrace{\underbrace{(72 - 24)} \cdot \frac{1}{4}}_{12} - 288 f''(8)$$

$$f''(8) = \frac{12}{\underset{12 \cdot 24}{288}} = \frac{1}{24}$$