REC-IV

#1: Consider the curve given by $y = \frac{16}{x} - x^2$.

Find the points where this curve has a horizontal tangent line.

m : the slope of the tangent line at the point $(a, f(a))$

$$\boxed{m = f'(a)}$$

$$m = 0 \Rightarrow \boxed{f'(a) = 0}$$

$$y = f(x) = \frac{16}{x} - x^2 = 16 \cdot x^{-1} - x^2$$

$$f'(x) = -16 \cdot x^{-2} - 2x$$

$$f'(x) = 0 \Leftrightarrow -\frac{16}{x^2} - 2x = 0 \Leftrightarrow \frac{-16 - 2x^3}{x^2} = 0.$$

$$\Leftrightarrow -16 - 2x^3 = 0 \Leftrightarrow x^3 = -8 \Leftrightarrow x = -2.$$

$$a = -2 \Rightarrow f(a) = \frac{16}{-2} - (-2)^2 = -12.$$

So, at the point $(-2, -12)$, the tangent line of $y = f(x)$ will be horizontal.

#2: Calculate the derivative of the given function using the definition of the derivative.

$$y = \frac{1}{\sqrt{1+x^2}} = f(x).$$

[The derivative of f : $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$].

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{\sqrt{1+x^2}\cdot\sqrt{1+(x+h)^2}\cdot h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{\sqrt{1+x^2}\sqrt{1+(x+h)^2}\cdot h} \cdot \frac{\left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}{\left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}$$

$$= \lim_{h\to 0} \frac{(\cancel{1}+\cancel{x^2}) - (\cancel{1}+\cancel{x^2}+2xh+h^2)}{h\cdot\sqrt{1+x^2}\sqrt{1+(x+h)^2}\left(\sqrt{1+x^2}+\sqrt{1+(x+h)^2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h} \cdot \sqrt{1+x^2} \cdot \sqrt{1+(x+h)^2} \cdot \left(\sqrt{1+x^2}+\sqrt{1+(x+h)^2}\right)}$$

$$= \frac{-2x}{2\left(\sqrt{1+x^2}\right)^3} = \frac{-x}{(1+x^2)^{3/2}}$$

#3: How should the function $f(x) = x^2 \cdot \sin(1/x)$ be defined at $x=0$, so that it is continuous at $x=0$? Is it then differentiable there?

$\left[f \text{ is cont. at } x=a \iff \lim_{x \to a} f(x) = f(a) \right]$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cdot \sin(1/x) = 0$$

$-1 \le \sin x \le 1 \quad \forall x \in \mathbb{R}$

$-1 \le \sin(1/x) \le 1 \quad \forall x \in \mathbb{R}$

$-c \quad 0 \quad c$

$-1 \le \sin(1/x) \le 1 \quad \forall x \in (-c, c)$

$x^2 > 0$

$-x^2 \le x^2 \cdot \sin(1/x) \le x^2 \quad \forall x \in (-c, c)$

$$\lim_{x\to 0} \underline{-x^2} \le \lim_{x\to 0} x^2 \cdot \sin(1/x) \le \lim_{x\to 0} \underline{x^2}$$

$$= 0 \qquad\qquad = 0.$$

$$\therefore \lim_{x\to 0} x^2 \cdot \sin(1/x) = 0.$$

by Squeeze Thm.

$$\text{Define } f(x) = \begin{cases} x^2 \cdot \sin(1/x) & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

Now, f is continuous at the pt $x=0$.

[Recall: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$]

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \cdot \sin(1/h) - 0}{h} = \lim_{h\to 0} h \cdot \sin(1/h) = 0$$

$-1 \le \sin h \le 1 \quad \forall h \in \mathbb{R}$

$-1 \le \sin(1/h) \le 1 \quad \forall h \in \mathbb{R}.$

$$-1 \le \sin(1/h) \le 1 \quad \forall h \in (0,c).$$

$$(h>0) \quad -h \le h\cdot\sin(1/h) \le h. \quad \forall h \in (0,c).$$

Take lim as $h \to 0^+$

$$\underbrace{\lim_{h\to 0^+} -h}_{=0} \le \lim_{h\to 0^+} h\cdot\sin(1/h) \le \underbrace{\lim_{h\to 0^+} h}_{=0}.$$

$\therefore$ By Squeeze Thm $\lim_{h\to 0^+} h\cdot\sin(1/h) = 0.$

Now, $-1 \le \sin(1/h) \le 1 \quad \forall h \in \mathbb{R}.$

$$-1 \le \sin(1/h) \le 1 \quad \forall h \in (-c,0)$$

$$(h<0) \quad h \le h\cdot\sin(1/h) \le -h. \quad \forall h \in (-c,0)$$

Take lim as $h \to 0^-$

$$\underbrace{\lim_{h\to 0^-} h}_{=0} \le \lim_{h\to 0^-} h\cdot\sin(1/h) \le \underbrace{\lim_{h\to 0^-} -h}_{=0.}$$

$\therefore \lim_{h\to 0^-} h\cdot\sin(1/h) = 0.$ by Squeeze Thm.

$\therefore \lim_{h\to 0} h\cdot\sin(1/h) = 0.$ Therefore, f is differentiable at $x=0$ and $f'(0)=0$.

#4: Let $g(x)$ be continuous at $x=a$ and consider the function $f(x) = (x-a)\cdot g(x)$.

Find $f'(a)$ in terms of g.

[Product Rule: $(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$.]

$$f'(x) = [(x-a)\cdot g(x)]' = \underbrace{(x-a)'}_{=1}\cdot g(x) + (x-a)\cdot g'(x)$$

$$\Rightarrow f'(x) = g(x) + (x-a)\cdot g'(x)$$

$$\Rightarrow f'(a) = g(a) + \underbrace{(a-a)}_{=0}\cdot g'(a) = \underline{\underline{g(a)}}.$$

#5: Given that $f(1)=2$, $f'(1)=1$, $g(1)=3$ and $g'(1)=4$. Calculate the followings:

(a) $\left.\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)+x}\right)\right|_{x=1}$.

[Quotient Rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$]

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Bigg|_{x=1} = \frac{f'(x)\cdot(g(x)+x) - f(x)\cdot(g(x)+x)'}{(g(x)+x)^2}\Bigg|_{x=1}$$

$$= \frac{f'(x)\cdot(g(x)+x) - f(x)\cdot(g'(x)+1)}{(g(x)+x)^2}\Bigg|_{x=1}$$

$$= \frac{\overset{=1}{\overline{f'(1)}}\cdot(\overset{=3}{\overline{g(1)}}+1) - \overset{=2}{\overline{f(1)}}\cdot\overset{=4}{\overline{(g'(1)+1)}}}{(\underbrace{g(1)}_{=3}+1)^2}$$

$$= \frac{4-10}{16} = -\frac{3}{8}$$

(b) $\frac{d}{dx}\left(f^2(x)\cdot g(x)\right)\Big|_{x=1}$

[Chain Rule: $[f(g(x))]' = f'(g(x))\cdot g'(x)$]

$$= \underbrace{\left(f^2(x)\right)'}_{*}\cdot g(x) + f^2(x)\cdot g'(x)\Big|_{x=1}$$

$$* = (f(x) \cdot f(x))' = f'(x) \cdot f(x) + f(x) \cdot f'(x)$$

by using product rule

$$= \underline{2 \cdot f'(x) \cdot f(x)}.$$

$$* = (x^2 \circ f(x))' = \underline{2f(x) \cdot f'(x)}.$$

by using chain rule.

$$f^2(x) = x^2 \circ f(x).$$

$$= 2f'(x) \cdot f(x) \cdot g(x) + f^2(x) \cdot g'(x) \Big|_{x=1}.$$

$$= 2\underbrace{f'(1)}_{=1} \cdot \underbrace{f(1)}_{=2} \cdot \underbrace{g(1)}_{=3} + \underbrace{f^2(1)}_{=2^2} \cdot \underbrace{g'(1)}_{4}.$$

$$= 12 + 16 = \underline{\underline{28}}.$$

(c) $\frac{d}{dx}(x^3 \cdot f(x)) \Big|_{x=1}$. Exercise.

#6: Find the derivative of the following functions:

(a) $f(x) = \sqrt{3x + \sqrt{2 + \sqrt{1-x}}}$

$$\left[\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} \cdot u' \right]$$

$$f'(x) = \frac{1}{2\underbrace{\sqrt{3x+\sqrt{2+\sqrt{1-x}}}}_{=f(x)}} \cdot \left(3x+\sqrt{2+\sqrt{1-x}}\right)'$$

$$= \frac{1}{2f(x)} \cdot \left(3 + \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \left(2+\sqrt{1-x}\right)'\right.$$

$$= \frac{1}{2f(x)} \left(3 + \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \frac{1}{2\sqrt{1-x}} \underbrace{(1-x)'}_{=-1} \right)$$

$$= \frac{1}{2f(x)} \left(3 - \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \frac{1}{2\sqrt{1-x}} \right)$$

(b) $g(x) = \left(\dfrac{1+\sin 3x}{3-2x} \right)^{-1}$.

$$g(x) = \frac{3-2x}{1+\sin 3x}$$

$$g'(x) = \frac{(3-2x)'(1+\sin 3x) - (3-2x).(1+\sin 3x)'}{(1+\sin 3x)^2}$$

$$= \frac{-2.(1+\sin 3x) - (3-2x).\cos 3x.3}{(1+\sin 3x)^2}$$

(c) $h(x) = \tan\left(\dfrac{\pi}{\sqrt{25-x^2}}\right)$ Exercise.

#7: (a) Suppose f is a differentiable function and $y = \dfrac{x}{4} - 3$ is an equation for the tangent line to the graph of $y=f(x)$ at the point $x=8$. If $g(x) = (f(x^3))^2$ find an equation for the tangent line to the graph of $y=g(x)$ at the point $x=2$

the equation for a tangent line at the pt $(a, f(a))$ is

$$y = f'(a) \cdot (x-a) + f(a).$$

$$y = \frac{1}{4} \cdot x - 3 = \overbrace{f'(8)}^{1/4} \cdot (x-8) + f(8)$$

$$\Rightarrow \boxed{f'(8) = \frac{1}{4}} \ \&$$

$$-3 = \frac{1}{4} \cdot (-8) + f(8).$$

$$\Rightarrow \boxed{f(8) = -1}$$

the tangent line of g at $x=2$,

$$y = \underline{g'(2)} \cdot (x-2) + g(2). \checkmark$$

$$\underline{g(2)} = (f(2^3))^2 = (f(8))^2 = (-1)^2 = \underline{1}.$$

$$\boxed{g'(x) = ((\underline{f(x^3)})^2)' = 2 \cdot f(x^3) \cdot f'(\underline{x^3}) \cdot 3x^2.}$$

$$g'(2) = 2 \cdot f(8) \cdot f'(8) \cdot 12 = 24 \cdot (-1) \cdot \frac{1}{4} = \underline{-6}.$$

$$\boxed{y = -6 \cdot (x-2) + 1}$$

(b) If $g''(2) = 0$ find $f''(8)$.

From part a,

$$g'(x) = \underbrace{6x^2 \cdot f(x^3)} \cdot \underbrace{f'(x^3)}$$

$$g''(x) = \underbrace{(6x^2 \cdot f(x^3))'} \cdot f'(x^3) + 6x^2 \cdot f(x^3) \cdot f''(x^3) \cdot 3x^2$$

$$= [12x \cdot f(x^3) + 6x^2 \cdot f'(x^3) \cdot 3x^2] \cdot f'(x^3) + 18x^4 \cdot f(x^3) \cdot f''(x^3)$$

$x=2$

$$0 = g''(2) = (24 \cdot \underbrace{f(8)}_{-1} + 18 \cdot 16 \cdot \underbrace{f'(8)}_{1/4}) \cdot \underbrace{f'(8)}_{1/4} + 18 \cdot 16 \cdot \underbrace{f(8)}_{-1} f''(8)$$

$$0 = \underbrace{(-24 + \underbrace{18 \cdot 4}_{72})}_{+48} \cdot \frac{1}{4} - 18 \cdot 16 \cdot f''(8)$$

$$\Rightarrow \quad f''(8) = \frac{+\cancel{12}\,2}{\cancel{18}_{3} \cdot 16} = \frac{1}{24}$$