REC - IV

#1: Consider the curve given by $y = \frac{16}{x} - x^2$. Find the points where this curve has a horizontal tangent line.

m: the slope of the tangent line at the point $(a, f(a))$

then $\boxed{m = f'(a)}$

$$y = f(x) = \frac{16}{x} - x^2$$

$$m = 0 \Rightarrow f'(a) = 0.$$

$$f(x) = 16 \cdot x^{-1} - x^2$$

$$f'(x) = -16 \cdot x^{-2} - 2x = 0 \Leftrightarrow \frac{-16}{x^2} - \frac{2x}{\underset{(x^2)}{1}} = 0$$

$$\frac{-16 - 2x^3}{x^2} = 0 \Leftrightarrow -16 - 2x^3 = 0$$

$$\Leftrightarrow x^3 = -8 \Leftrightarrow x = -2.$$

Therefore, $a = -2$, $\Rightarrow f(a) = \frac{16}{-2} - (-2)^2 = -12.$

So, at the pt $(-2, -12)$, $y = f(x)$ will have a horizontal tangent line.

#2: Calculate the derivative of the given function using the definition of the derivative

$$y = \frac{1}{\sqrt{1+x^2}} = f(x)$$

[Recall: the derivative of f $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$]

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{h \cdot \sqrt{1+(x+h)^2} \cdot \sqrt{1+x^2}} \cdot \frac{\left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}{\left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}$$

$$= \lim_{h \to 0} \frac{(\cancel{1} + \cancel{x^2}) - (\cancel{1} + \cancel{x^2} + 2xh + h^2)}{h \sqrt{1+(x+h)^2} \cdot \sqrt{1+x^2} \cdot \left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h} \cdot \sqrt{1+(x+h)^2} \cdot \sqrt{1+x^2} \cdot \left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}$$

$$= \frac{-2x}{2\left(\sqrt{1+x^2}\right)^3} = \frac{-x}{(1+x^2)^{3/2}}$$

#3: How should the function $f(x) = x^2 \cdot \sin(1/x)$ be defined at $x=0$ so that it is continuous at $x=0$? Is it then differentiable there?

$$\left[f \text{ is cont. at } x=a \iff \lim_{x \to a} f(x) = f(a) \right].$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cdot \sin(1/x) = 0$$

Squeeze Thm.

$$-1 \leq \sin(x) \leq 1 \quad \forall x \in \mathbb{R}.$$

$$-1 \leq \sin(1/x) \leq 1 \quad \forall x \in \mathbb{R}$$

$$-1 \leq \sin(1/x) \leq 1 \quad \forall x \in (-c, c).$$

$-c \quad 0 \quad c$

$(x^2 \geq 0)$

$$-x^2 \leq x^2 \sin(1/x) \leq x^2.$$

$$\underbrace{\lim_{x \to 0} (-x^2)}_{=0} \leq \lim_{x \to 0} x^2 \sin(1/x) \leq \underbrace{\lim_{x \to 0} x^2}_{=0}.$$

$$\therefore \lim_{x \to 0} x^2 \sin(1/x) = 0$$

by Squeeze Thm.

Now, define $f(x) = \begin{cases} x^2 \cdot \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Thus way, f becomes continuous at $x=0$.

[Recall: $f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$]

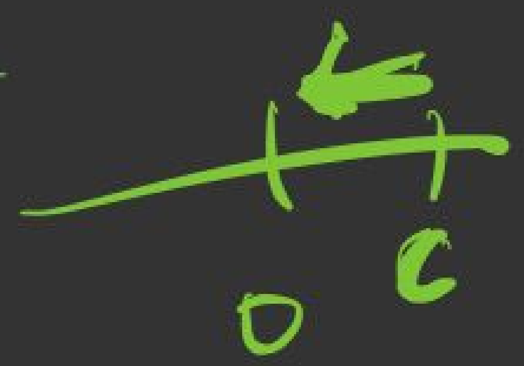
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \cdot \sin(1/h) - 0}{h} = \lim_{h \to 0} h \cdot \sin(1/h)$$

$= 0$ by Squeeze Thm.

$-1 \leq \sin h \leq +1 \quad \forall x \in \mathbb{R}$.

$-1 \leq \sin(1/h) \leq 1 \quad \forall x \in \mathbb{R}$.

For $h \geq 0$. 

$-1 \leq \sin(1/h) \leq 1 \quad \forall h \in (0,c)$.

$-h \leq h \cdot \sin(1/h) \leq h. \quad \forall h \in (0,c)$

$$\underbrace{\lim_{h\to 0^+} -h}_{=0} \le \lim_{h\to 0^+} h\cdot\sin(1/h) \le \underbrace{\lim_{h\to 0^+} h}_{=0}.$$

$$\therefore \lim_{h\to 0^+} h\cdot\sin(1/h) = 0 \text{ by Squeeze Thm.}$$

For $h \le 0$

$$-1 \le \sin(1/h) \le 1 \quad \forall h \in (-c, 0).$$

$$h \le h\cdot\sin(1/h) \le -h \quad \forall h \in (-c, 0).$$

$$\underbrace{\lim_{h\to 0^-} h}_{=0} \le \lim_{h\to 0^-} h\cdot\sin(1/h) \le \underbrace{\lim_{h\to 0^-} -h}_{=0}.$$

$$\therefore \lim_{h\to 0^-} h\cdot\sin(1/h) = 0 \text{ by Squeeze Thm.}$$

Hence, f is differentiable at $x=0$ and

$f'(0) = 0$.

#4: Let $g(x)$ be continuous at $x=a$ and consider the function $f(x) = (x-a) \cdot g(x)$

Find $f'(a)$ in terms of g.

[Product Rule: $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$]

$$f'(x) = [(x-a) \cdot g(x)]' = \underbrace{(x-a)'}_{=1} \cdot g(x) + (x-a) \cdot g'(x)$$

$$f'(x) = g(x) + (x-a) \cdot g'(x)$$

$$f'(a) = g(a) + \underbrace{(a-a)}_{=0} \cdot g'(a) = g(a)$$

#5: Given that $f(1) = 2$, $f'(1) = 1$, $g(1) = 3$ and $g'(1) = 4$. Calculate the following:

(a) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)+x}\right)\Bigg|_{x=1}$

[Quotient Rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$]

$$= \left. \frac{f'(x)\cdot(g(x)+x) - f(x)\cdot(g(x)+x)'}{(g(x)+x)^2} \right|_{x=1}$$

$$= \left. \frac{f'(x)\cdot(g(x)+x) - f(x)(g'(x)+1)}{(g(x)+x)^2} \right|_{x=1}$$

$$= \frac{\overbrace{f'(1)}^{=1}\cdot(\overbrace{g(1)}^{=3}+1) - \overbrace{f(1)}^{=2}\cdot(\overbrace{g'(1)}^{4}+1)}{(\underbrace{g(1)}_{=3}+1)^2}$$

$$= \frac{4-10}{16} = \underline{\underline{-\frac{3}{8}}}$$

b) $\left. \frac{d}{dx}\left(f^2(x)\cdot g(x)\right) \right|_{x=1}$

Chain Rule:

$\rightarrow$ $\left[(f(g(x))' = f'(g(x))\cdot \underline{g'(x)} \right]$

$\underline{f^2(x)} = x^2 \circ \underline{f(x)}$

$(f^2(x))' = 2f(x)\cdot f'(x).$

OR. you can apply product rule.

$$\left(f^2(x)\right)' = \left(f(x)\cdot f(x)\right)' = f'(x)\cdot f(x) + f(x)\cdot f'(x).$$
$$= 2\cdot f(x)\cdot f'(x).$$

$$= \left(f^2(x)\right)'\cdot g(x) + f^2(x)\cdot g'(x)\Big|_{x=1}$$

$$= 2\, f(x)\cdot f'(x)\cdot g(x) + f^2(x)\cdot g'(x)\Big|_{x=1}$$

$$= 2\cdot \underbrace{f(1)}_{2}\cdot \underbrace{f'(1)}_{1}\cdot \underbrace{g(1)}_{3} + \underbrace{f^2(1)}_{2^2}\cdot \underbrace{g'(1)}_{4}.$$

$$= 12 + 16 = \underline{\underline{28}}.$$

(c) $\frac{d}{dx}\left(x^3\cdot f(x)\right)\Big|_{x=1}$ Exercise.

#6: Find the derivative of the following functions:

(a) $f(x) = \sqrt{3x + \sqrt{2 + \sqrt{1-x}}}$ $\quad (\sqrt{u})' = \frac{1}{2\sqrt{u}}\cdot u'.$

$$f'(x) = \frac{1}{2\underbrace{\sqrt{3x+\sqrt{2+\sqrt{1-x}}}}_{=f(x)}} \cdot \left(3x+\sqrt{2+\sqrt{1-x}}\right)'$$

$$= \frac{1}{2f(x)} \cdot \left(3x+\sqrt{2+\sqrt{1-x}}\right)'$$

$$= \frac{1}{2f(x)} \cdot \left(3 + \frac{1}{2\sqrt{2+\sqrt{1-x}}} \cdot \left(2+\sqrt{1-x}\right)'\right)$$

$$= \frac{1}{2f(x)} \left(3 + \frac{1}{2\sqrt{2+\sqrt{1-x}}} \frac{1}{2\sqrt{1-x}}(-1)\right)$$

(b) $g(x) = \left(\dfrac{1+\sin 3x}{3-2x}\right)^{-1}$.

$$g(x) = \frac{3-2x}{1+\sin 3x}$$

$$g'(x) = \frac{(3-2x)' \cdot (1+\sin 3x) - (3-2x) \cdot (1+\sin 3x)'}{(1+\sin 3x)^2}$$

$$= \frac{-2 \cdot (1+\sin 3x) - (3-2x) \cdot \cos 3x \cdot 3}{(1+\sin 3x)^2}$$

(c) $h(x) = \tan \dfrac{\pi}{\sqrt{25-x^2}}$ Exercise.

#7: (a) Suppose f is a differentiable function and $y = \frac{x}{4} - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point $x = 8$. If $g(x) = (f(x^3))^2$ find an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 2$.

$(a, f(a))$ $\quad m = f'(a)$.

$$\boxed{y = f'(a).(x-a) + f(a)}$$ the eqn for the tangent line.

$$y = \frac{1}{4}x - 3 = \frac{1}{4}(x-8) + f(8)$$

$$\boxed{f'(8) = \frac{1}{4}}$$

$$-2 + f(8) = -3.$$

$$\Rightarrow \boxed{f(8) = -1}$$

$$g'(x) = ((f(x^3))^2)' = 2 . f(x^3) . f'(x^3) . 3x^2$$

$$g'(2) = 2 . f(8) . f'(8) . 12$$

$$= 2 . \cancel{12}^{3} . (-1) . \frac{1}{\cancel{4}} = -6$$

$$g(2) = (f(8))^2 = (-1)^2 = 1.$$

$$y = g'(2)(x-2) + g(2).$$

$$\Rightarrow y = -6(x-2) + 1.$$

(b) If $g''(2) = 0$, find $f''(8)$.

From part a,

$$g'(x) = \underbrace{6x^2 \cdot f(x^3)} \cdot \underline{f'(x^3)}$$

$$g''(x) = (\underbrace{6x^2 f(x^3)})' f'(x^3) + 6x^2 \cdot f(x^3)\, f''(x^3) \cdot 3x^2$$

$$= \left[12x\, f(x^3) + 6x^2 \cdot f'(x^3) \cdot 3x^2\right] f'(x^3) + 18x^4 \cdot \underline{f(x^3)} \cdot f''(x^3)$$

$$0 = g''(2) = \left[24 \cdot f(8) + 18 \cdot 16 \cdot f'(8)\right] \cdot f'(8)$$

$$+ 18 \cdot 16 \cdot f''(8) \cdot f(8)$$

$$0 = \left(\underbrace{24 \cdot (-1)}_{-24} + \underbrace{18 \cdot \cancel{16}^{4} \cdot \frac{1}{\cancel{4}}}_{72}\right)\frac{1}{4} + \underline{18 \cdot 16 \cdot (-1) f''(8)}$$

$$48$$

$$12$$

$$\Rightarrow \quad f''(8) = \frac{\cancel{12}^{3}}{\cancel{18}_{6} \cdot \cancel{16}_{4}} = \frac{1}{24}.$$