REC – II

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#1: Sketch the graph of the following function and evaluate the limits

$$f(x) = \begin{cases} 2-x & \text{if } x<-1 \\ x & \text{if } -1 \le x < 1 \\ (x-1)^2 & \text{if } x \geq 1. \end{cases}$$

a) $\lim\limits_{x \to -1} f(x) = ?$

$\lim\limits_{x \to a} f(x) = L \iff \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L.$

$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} 2-x = 3.$

$x < -1.$

-1

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} x = -1.$

$x > -1.$

-1

$\neq.$

$\lim\limits_{x \to -1} f(x)$ does not exist!

DNE

b) $\lim_{x \to 0} f(x) = ?$

$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} x = 0 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x = 0 \end{aligned}\right\} =$$

$$\lim_{x \to 0} f(x) = 0$$

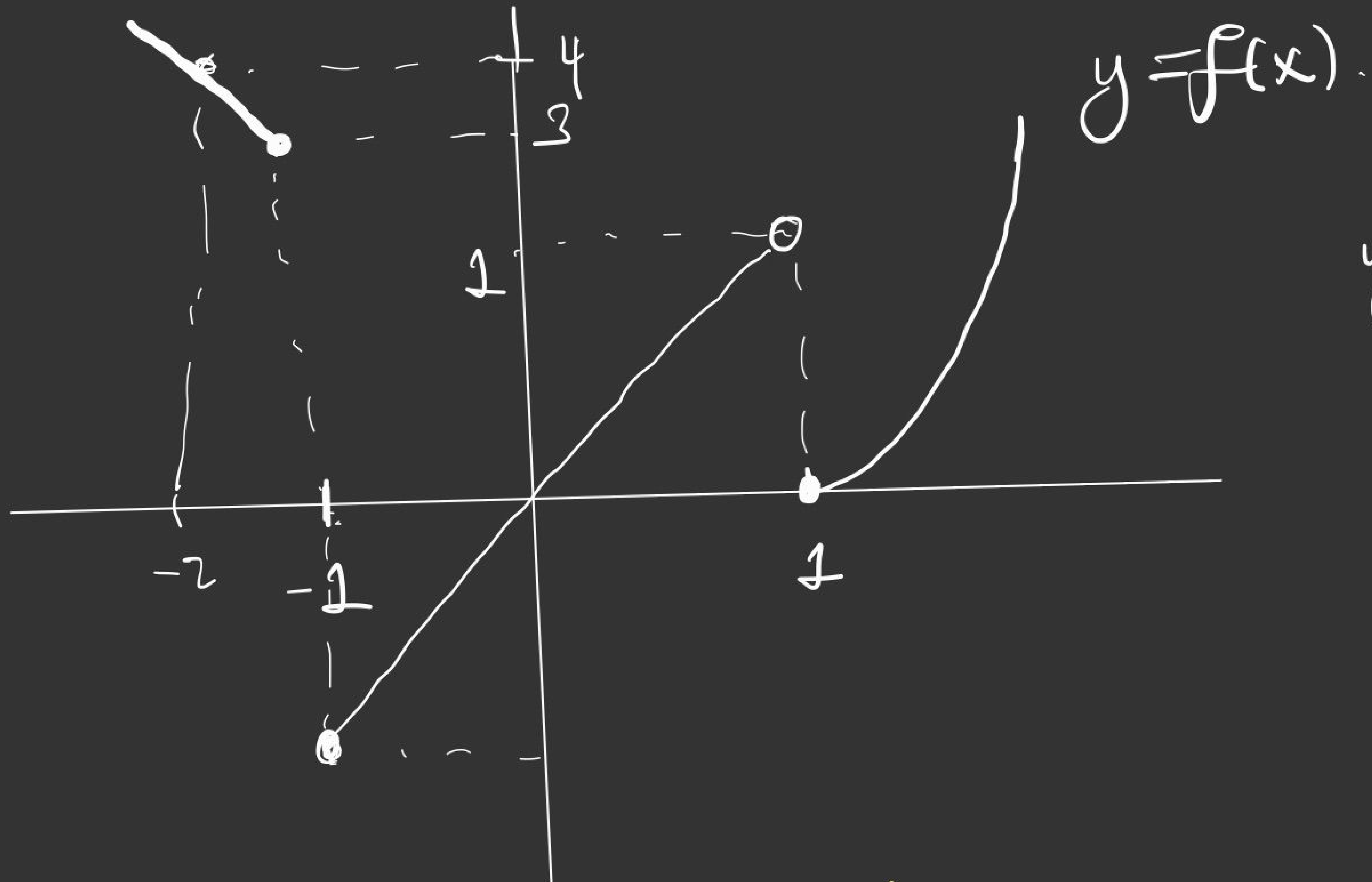
c) $\lim_{x \to 1} f(x) = ?$

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (x-1)^2 = 0. \end{aligned}\right\} \neq .$$

1 $x<1$

1 $x>1$

$\lim_{x \to 1} f(x)$ DNE!



$y = 2 - x.$

$x = -1 \Rightarrow y = ?$

$x = -2 \Rightarrow y = 4$

#2: Let $f(x) = \dfrac{|x-2|}{x-2}$.

(a) $\lim\limits_{x \to 0} f(x) = ?$

$$\lim_{x \to 0} \frac{|x-2|}{x-2} = \frac{|-2|}{-2} = \underline{\underline{-1}}$$

(b) $\lim\limits_{x \to 2} f(x) = ?$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-x+2}{x-2} = -1.$$

$\xrightarrow{\quad} 2$ $\quad x < 2.$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1.$$

$x > 2.$

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$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x).$$

Therefore, $\lim_{x\to 2} f(x)$ DNE!

#3: Evaluate the limit if it exists.

a) $$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}\cdot(x^2+x+1)}{\cancel{(x-1)}(x+1)}$$

$$= \frac{3}{2}$$

(b) $$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{(\sqrt{1+h})^2 - 1}{h\cdot(\sqrt{1+h}+1)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)} = \frac{1}{2}$$

$$(c) \lim_{t \to 0} \frac{(3+t)^{-1} - 3^{-1}}{t}$$

$$= \lim_{t \to 0} \frac{\frac{1}{3+t}\left(\frac{3}{3}\right) - \frac{1}{3}\left(\frac{3+t}{3+t}\right)}{t} = \lim_{t \to 0} \frac{\frac{3-(3+t)}{3(3+t)}}{t}$$

$$= \lim_{t \to 0} \frac{-\cancel{t}}{3\cdot(3+t)} \cdot \frac{1}{\cancel{t}} = -\frac{1}{9}$$

$$(d) \lim_{x \to 1} \frac{\sqrt{x} - x^2}{1-\sqrt{x}} = \lim_{x \to 1} \frac{\sqrt{x} - (\sqrt{x})^4}{1-\sqrt{x}}$$

$$= \lim_{x \to 1} \frac{\sqrt{x}\left(1-(\sqrt{x})^3\right)}{1-\sqrt{x}}$$

$$= \lim_{x \to 1} \frac{\sqrt{x}\cdot\cancel{(1-\sqrt{x})}\cdot(1+\sqrt{x}+x)}{\cancel{1-\sqrt{x}}}$$

$$= 3$$

#4: Let $f(x) = x - \lfloor x \rfloor$

[$g(x) = \lfloor x \rfloor$ → the greatest integer function

It gives the largest integer less than or equal to x.]

a) If n is an integer, evaluate

(i) $\lim_{x \to n^-} f(x)$

(ii) $\lim_{x \to n^+} f(x)$

(i) $\lim_{x \to n^-} f(x) = \lim_{x \to n^-} x - \lfloor x \rfloor = n - (n-1)$

$x \to n$, $\lfloor x \rfloor \to n-1$.

$= \underline{\underline{1}}$.

n $\quad x < n$.

(ii) $\lim_{x \to n^+} f(x) = \lim_{x \to n^+} x - \lfloor x \rfloor = \underline{\underline{0}}$.

$x \to n$, $\lfloor x \rfloor \to n$

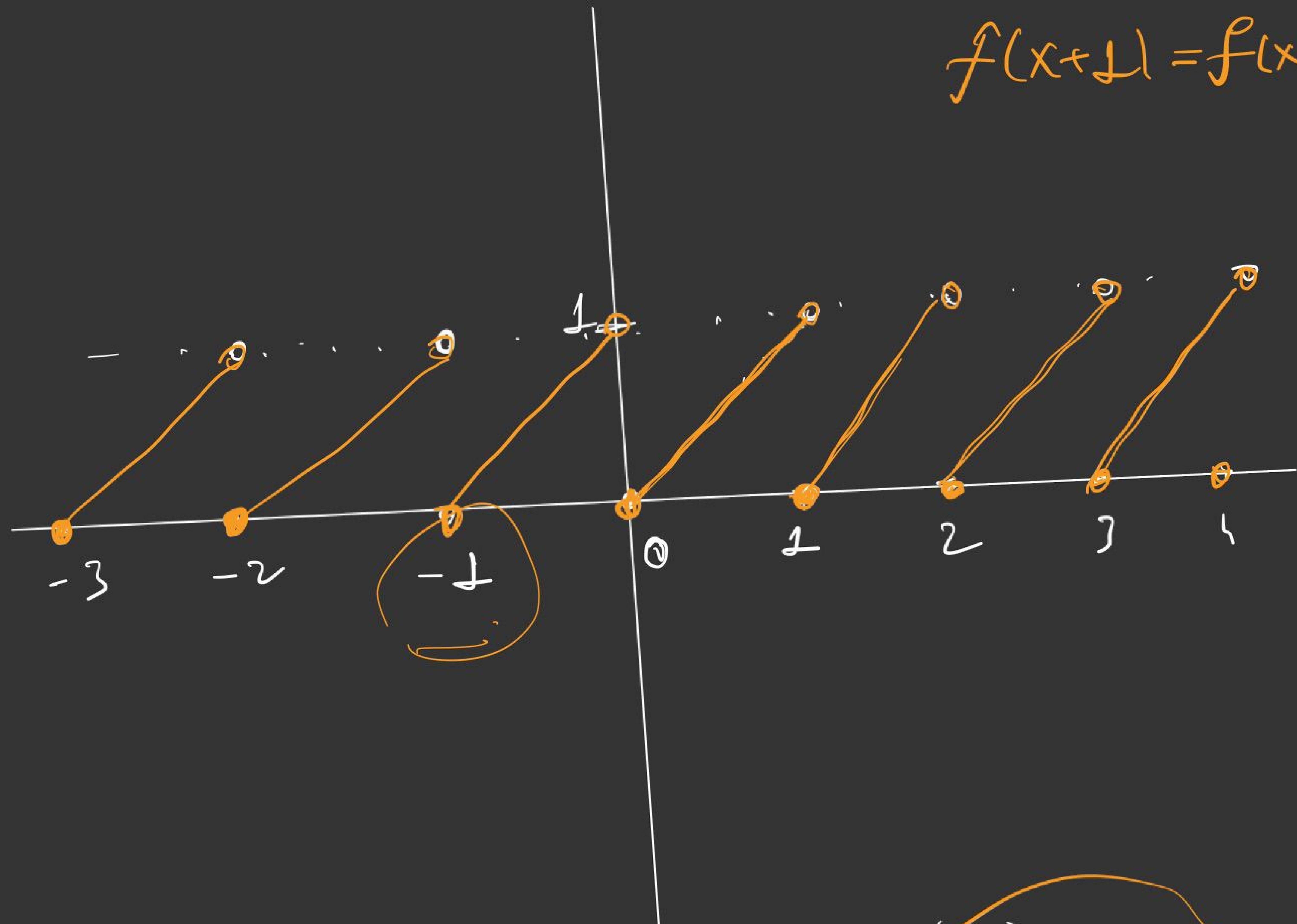
n $\quad x > n$.

(b) Sketch the graph of f.

$f(n) = n - \lfloor n \rfloor = n - n = 0$.

$n \in \mathbb{Z}$.

$f(x+1) = f(x).$



Take $0 \le x < 1$

$y = f(x) = x - \lfloor x \rfloor$

$y = x - 0$

$y = x.$

Take $1 \le x < 2$

$y = f(x) = x - \lfloor x \rfloor.$

$y = x - 1.$

Take $-1 \le x < 0.$

$y = f(x) = x - \lfloor x \rfloor$

$= x - (-1) = x + 1.$

(c) For what values of a, does $\lim_{x \to a} f(x)$ exist?

For integer values n,

$$\lim_{x \to n^-} f(x) = 1 \ \& \ \lim_{x \to n^+} f(x) = 0.$$

$$\neq.$$

$$\lim_{x \to n} f(x) \text{ DNE!}$$

$n \in \mathbb{Z}$.

$a \in \mathbb{R} \setminus \mathbb{Z}$.

$$\lim_{x \to a^-} f(x) = a - \lfloor a \rfloor = \lim_{x \to a^+} f(x)$$

Hence, we must have $\boxed{a \in \mathbb{R} \setminus \mathbb{Z}}$.

#5: If $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \underbrace{\lfloor x \rfloor}_{1} + \underbrace{\lfloor -x \rfloor}_{-2} = 1 - 2 = -1.$$

$\boxed{x < 2} \Rightarrow \boxed{-x > -2}$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \underbrace{\lfloor x \rfloor}_{2} + \underbrace{\lfloor -x \rfloor}_{-3} = -1.$$

$\boxed{x > 2} \Rightarrow \boxed{-x < -2}$

-3 $\quad$ -2

$$\boxed{\lim_{x \to 2} f(x) = -1.}$$

$$f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 + (-2) = \underline{\underline{0}}.$$

$\neq$

#6: Evaluate $\lim_{x \to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$

change variable $y = 3-x$ $\underset{2}{\downarrow}$ $x \to 2 \Rightarrow y \to 1$.

$$= \lim_{y \to 1} \frac{\sqrt{y+3}-2}{\sqrt{y}-1} = \lim_{y \to 1} \frac{\sqrt{y+3}-2}{\sqrt{y}-1} \cdot \frac{\sqrt{y+3}+2}{\sqrt{y+3}+2}$$

$$= \lim_{y \to 1} \frac{y+3-4 \quad y-1}{(\sqrt{y}-1)\cdot(\sqrt{y+3}+2)}$$

$$\lim_{y \to 1} \frac{\cancel{(\sqrt{y}-1)}(\sqrt{y}+1)}{\cancel{(\sqrt{y}-1)}(\sqrt{y+3}+2)}$$

$$= \frac{2}{4} = \frac{1}{2}.$$

#7 : Prove that

(a) $\lim_{x \to 0} x^4 \cdot \cos(2/x) = 0$.

$-a \quad 0 \quad a$

[Squeeze Thm: Suppose that $f(x) \leq g(x) \leq h(x)$ $\forall x \in I$ (I is an open interval containing a)

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x) \Rightarrow \lim_{x \to a} g(x) = L$.]

We know that $-1 \leq \cos x \leq 1$. $\forall x \in \mathbb{R}$.

$\forall x \in (-a, a)$

$$-1 \leq \cos(2/x) \leq 1 \quad \forall x \in (-a, a).$$

$\downarrow$ multiply ineq. by x^4 $(x^4 \geq 0)$

$$-x^4 \leq x^4 \cos(2/x) \leq x^4 \quad \forall x \in (-a, a)$$

$\downarrow$ Take lim as $x \to 0$ of every func. in the inequality

$$\underbrace{\lim_{x \to 0} -x^4}_{0} \leq \lim_{x \to 0} x^4 \cdot \cos(2/x) \leq \underbrace{\lim_{x \to 0} x^4}_{0}, \quad \forall x \in (-a, a)$$

$\therefore$ (Hence) $\lim\limits_{x\to 0} x^4\cos(2/x) = 0$

by Squeeze Thm. □

(b) $\lim\limits_{h\to 0^+} \sqrt{h}\cdot e^{\sin(\pi/h)} = 0$.

$$-1 \le \sin h \le 1 \quad \forall h\in\mathbb{R} \Rightarrow \forall h\in(0,a)$$

$$-1 \le \sin(\pi/h) \le 1 \quad \forall h\in(0,a).$$

$\downarrow$ Take the power of e.

$$e^{-1} \le e^{\sin(\pi/h)} \le e \quad \forall h\in(0,a)$$

$$\sqrt{h}\cdot e^{-1} \le \sqrt{h}\cdot e^{\sin(\pi/h)} \le \sqrt{h}\cdot e \quad \forall h\in(0,a)$$

$\downarrow$ Take lim as $h\to 0^+$.

$$\underbrace{\lim_{h\to 0^+}\sqrt{h}\cdot e^{-1}}_{0} \le \underbrace{\lim_{h\to 0^+}\sqrt{h}\cdot e^{\sin(\pi/h)}} \le \underbrace{\lim_{h\to 0^+}\sqrt{h}\cdot e}_{0}.$$

$$\therefore \lim_{h\to 0^+} \sqrt{h} \cdot e^{\sin(\pi/h)} = 0$$

by Squeeze Thm

$\square$

#8: Evaluate $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-2}$

∞

∞

∞

$= \underline{\underline{0}}$

#9: Evaluate $\displaystyle\lim_{x\to\infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1} \right)$.

$(x-1) \quad (x+1)$

$$= \lim_{x\to\infty} \frac{x^2 (x-1) - x^2 (x+1)}{(x-1)(x+1)}$$

$$= \lim_{x\to\infty} \frac{-2x^2}{x^2-1} = \lim_{x\to\infty} \frac{\cancel{x^2}(-2)}{\cancel{x^2}(1 - 1/x^2)}$$

$\downarrow 0$

$\downarrow 1$

$= \underline{\underline{-2}}.$

#8: (revisited).

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-2}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2(1-2/x)}-2}$$

$$= \lim_{x\to\infty} \frac{1}{x\cdot\sqrt{1-2/x}-2\cdot\frac{x}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\underset{\downarrow}{\underset{\infty}{x}}\underbrace{\left(\sqrt{1-\underset{\downarrow\,0}{2/x}}-\underset{\downarrow\,0}{2/x}\right)}_{\downarrow\,1}} = 0.$$