REC – II

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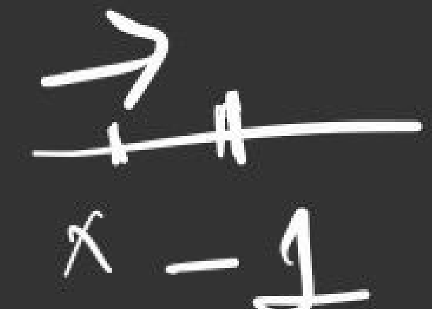
#1: Sketch the graph of the following function and evaluate the limits

$$y = f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ (x-1)^2 & \text{if } x \geq 1. \end{cases}$$

a) $\lim\limits_{x \to -1} f(x) = ?$

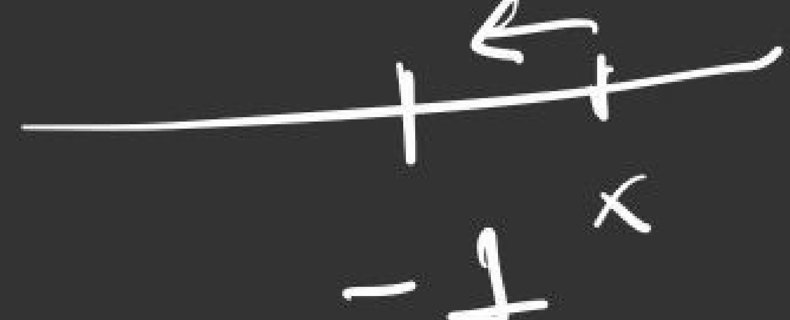
$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 2-x = 2-(-1) = 3.$$



$x < -1$.

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1.$$



$x > -1$.

$$\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$$

Therefore, $\lim_{x\to -1} f(x)$ does not exist!

DNE.

(b) $\lim_{x\to 0} f(x) = ?$

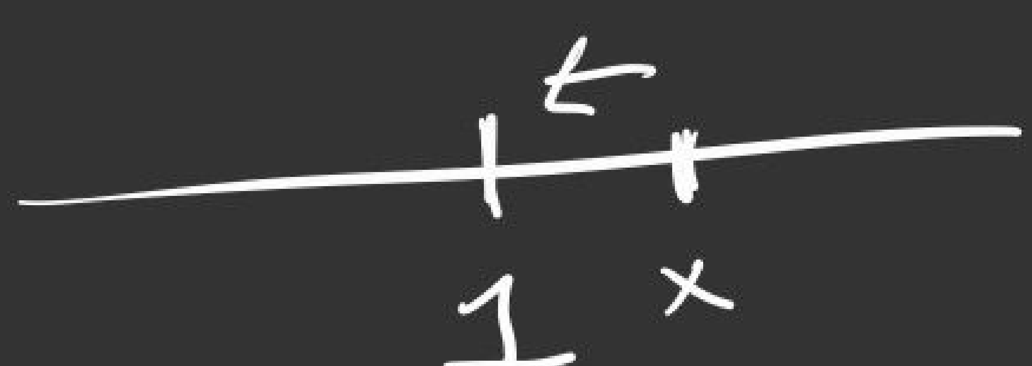
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x = \underline{\underline{0}}.$$

(c) $\lim_{x\to 1} f(x) = ?$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1.$$

$x < 1$.

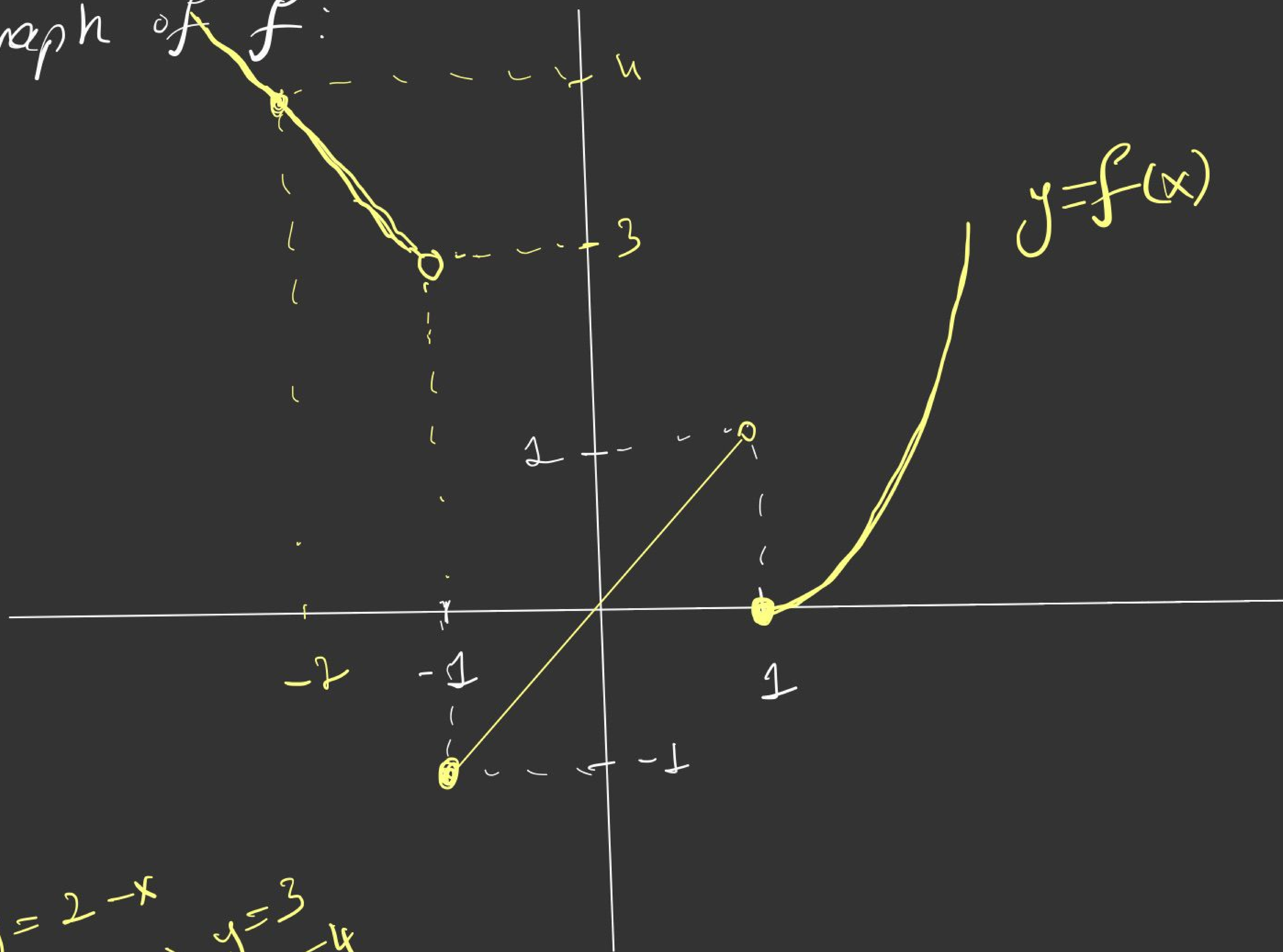
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0.$$



$x > 1$.

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) \Rightarrow \lim_{x\to 1} f(x) \text{ DNE!}$$

Graph of f:



$y = 2 - x$

$x = -1 \Rightarrow y = 3$

$x = -2 \Rightarrow y = 4$

#2: Let $f(x) = \dfrac{|x-2|}{x-2}$.

a) $\lim\limits_{x \to 0} f(x) = ?$

$$= \lim_{x \to 0} \frac{|x-2|}{x-2} = \frac{|-2|}{-2} = -1.$$

b) $\lim_{x \to 2} f(x) = ?$ $\left[\lim_{x \to a} c = c \right]$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{\overset{-(x-2)}{-x+2}}{\cancel{x-2}} = -1$$

$x < 2$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = 1$$

$x > 2$

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$ So, $\lim_{x \to 2} f(x)$ DNE!

#3: Evaluate the limit if it exists

a) $$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}$$

$$= \frac{3}{2}$$

(b) $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$

$= \lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h} \cdot \dfrac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$

$= \lim_{h\to 0} \dfrac{(\sqrt{1+h})^2 - 1}{h\cdot(\sqrt{1+h}+1)}$

$= \lim_{h\to 0} \dfrac{\underbrace{1+h-1}_{=\cancel{h}}}{\cancel{h}\cdot(\sqrt{1+\underset{0}{h}}+1)} = \dfrac{1}{2}$

(c) $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$

$= \lim_{x\to 1} \dfrac{\sqrt{x}-(\sqrt{x})^4}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{\sqrt{x}\cdot(1-(\sqrt{x})^3)}{1-\sqrt{x}}$

$= \lim_{x\to 1} \dfrac{\sqrt{x}\cdot\cancel{(1-\sqrt{x})}(1+\sqrt{x}+x)}{\cancel{1-\sqrt{x}}}$ $\quad 1-t^3 = (1-t)(1+t+t^2)$

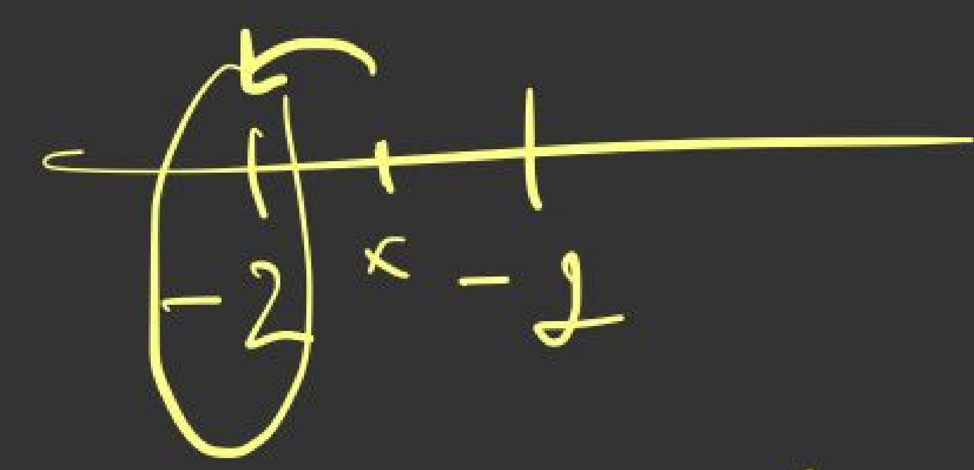
$t = \sqrt{x}$

$= 3$

#4: Let $y = f(x) = x - \lfloor x \rfloor$.

$g(x) = \lfloor x \rfloor \rightarrow$ the greatest integer function)

It gives the largest integer less than or equal to x.



$-2 < x < -1$

$\lfloor x \rfloor = -2$.

a) If n is an integer, evaluate $\boxed{n \in \mathbb{Z}.}$

(i) $\lim\limits_{x \to n^-} f(x)$

(ii) $\lim\limits_{x \to n^+} f(x)$

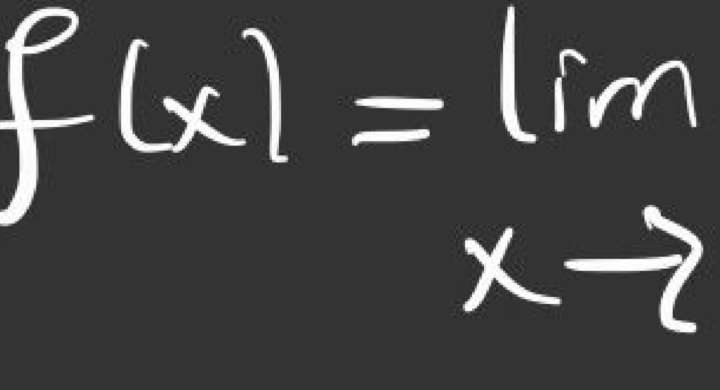
(i) $\lim\limits_{x \to n^-} f(x) = \lim\limits_{x \to n^-} \underbrace{x}_{n} - \underbrace{\lfloor x \rfloor}_{n-1} = n - (n-1)$

$= 1$.

$\boxed{x < n}$

(ii) $\lim\limits_{x \to n^+} f(x) = \lim\limits_{x \to n^+} \underbrace{x}_{n} - \underbrace{\lfloor x \rfloor}_{n} = n - n$

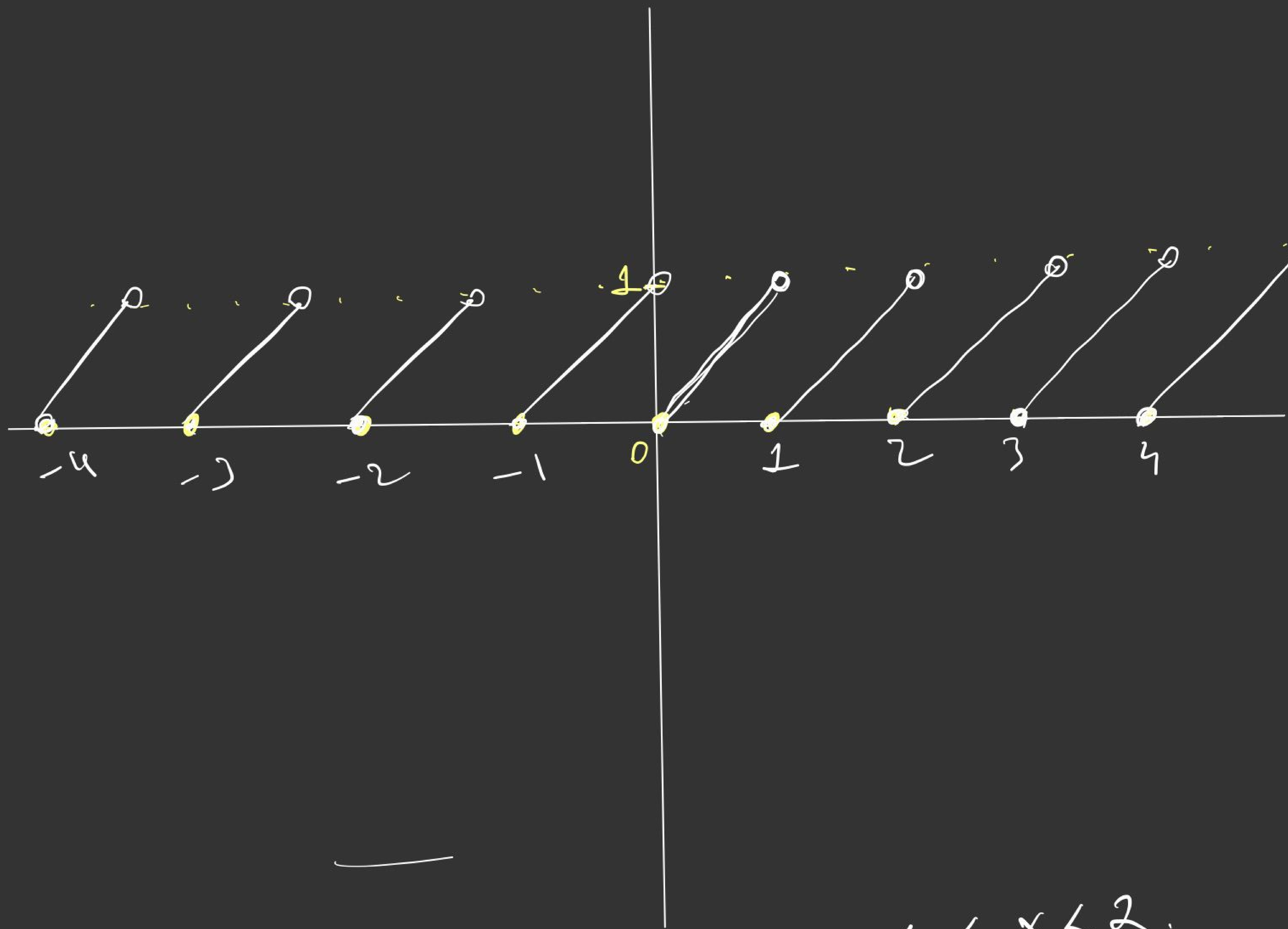
$= 0$.



$x > n$.

b) Sketch the graph of f.

$$f(n) = n - \underbrace{\lfloor n \rfloor}_{=n} = 0.$$



Take $0 \leq x < 1$.

$y = f(x) = x - \lfloor x \rfloor$.

$y = x - 0$.

$y = x$.

Take $1 \leq x < 2$.

$y = f(x) = x - \lfloor x \rfloor$

$y = x - 1$.

c) For what values of a does $\lim_{x \to a} f(x)$ exist?

when $n \in \mathbb{Z}$,

$$\lim_{x \to n^-} f(x) = 1 \ \& \ \lim_{x \to n^+} f(x) = 0.$$

$$\lim_{x \to n^-} f(x) \neq \lim_{x \to n^+} f(x)$$

Therefore, $\lim_{x \to n} f(x)$ does not exist.

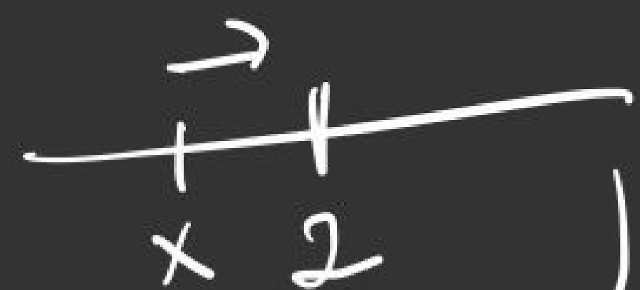
Take $a \in \mathbb{R} \setminus \mathbb{Z}$. Then,

$$\rightarrow \lim_{x \to a^-} f(x) = a - \lfloor a \rfloor = \lim_{x \to a^+} f(x).$$

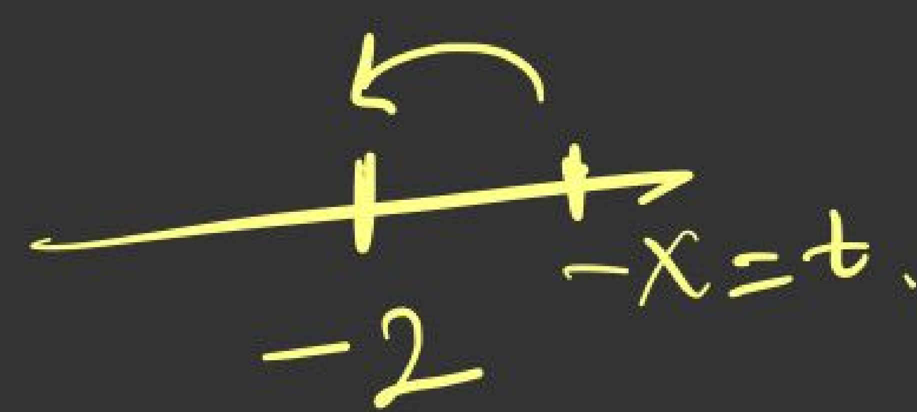
Therefore, $\lim_{x \to a} f(x)$ exists if $a \in \mathbb{R} - \mathbb{Z}$.

#5: If $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \underbrace{\lfloor x \rfloor}_{=1} + \underbrace{\lfloor -x \rfloor}_{-2} = 1 + (-2) = -1$$

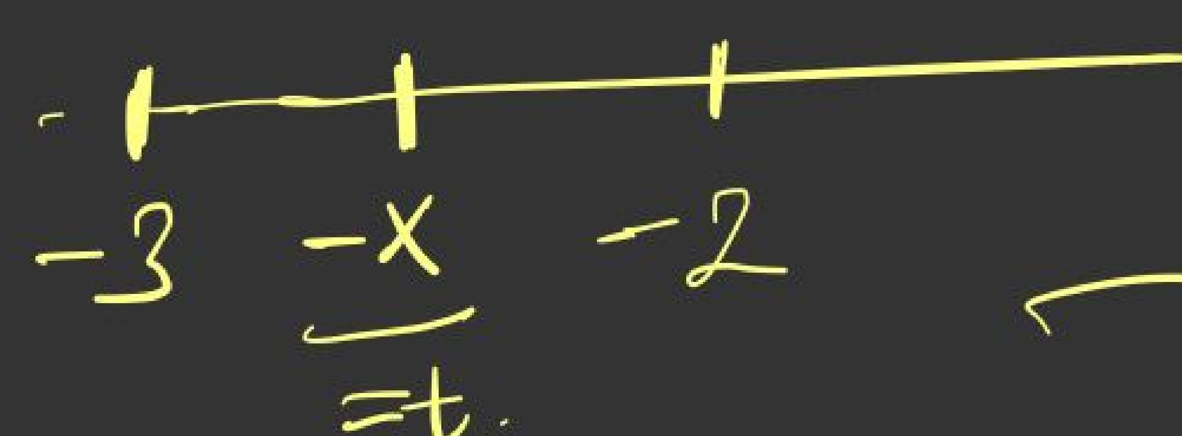


$\boxed{x < 2} \Rightarrow \boxed{-x > -2}$



$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \underbrace{\lfloor x \rfloor}_{2} + \underbrace{\lfloor -x \rfloor}_{-3} = 2 + (-3) = -1$$

$\boxed{x > 2} \Rightarrow \boxed{-x < -2}$



$$\lim_{x \to 2} f(x) = -1$$

$\neq$

$$f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 + (-2) = 0$$

#6: Evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$

change the variable

$$y = 3-x$$
$$y \to 1, \quad x \to 2$$

$$= \lim_{y\to 1} \frac{\sqrt{y+3}-2}{\sqrt{y}-1}$$

$$= \lim_{y\to 1} \frac{\sqrt{y+3}-2}{\sqrt{y}-1} \cdot \frac{\sqrt{y+3}+2}{\sqrt{y+3}+2}$$

$$= \lim_{y\to 1} \frac{\left(\sqrt{y+3}\right)^2-2^2}{\left(\sqrt{y}-1\right)\left(\sqrt{y+3}+2\right)}$$

$$= \lim_{y\to 1} \frac{y-1}{\left(\sqrt{y}-1\right)\left(\sqrt{y+3}+2\right)}$$

$$y-1 = \cancel{(\sqrt{y}-1)}(\sqrt{y}+1).$$

$$\frac{\cancel{(\sqrt{y}-1)}(\sqrt{y}+1)}{\cancel{(\sqrt{y}-1)}(\sqrt{y+3}+2)}$$

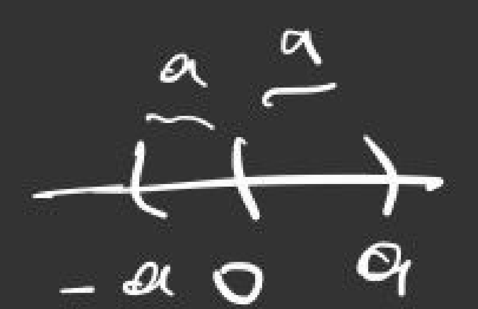
$$= \lim_{y\to 1} \frac{\sqrt{y}+1}{\sqrt{y+3}+2} = \frac{2}{4} = \underline{\underline{\frac{1}{2}}}.$$

#7: Prove that

(a) $\lim_{x \to 0} x^4 \cdot \cos(2/x) = 0.$

Recall: Squeeze Thm

Suppose that $f(x) \leq g(x) \leq h(x)$ $\forall x \in I$ where I is an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$. Then $\lim_{x \to a} g(x) = L$.



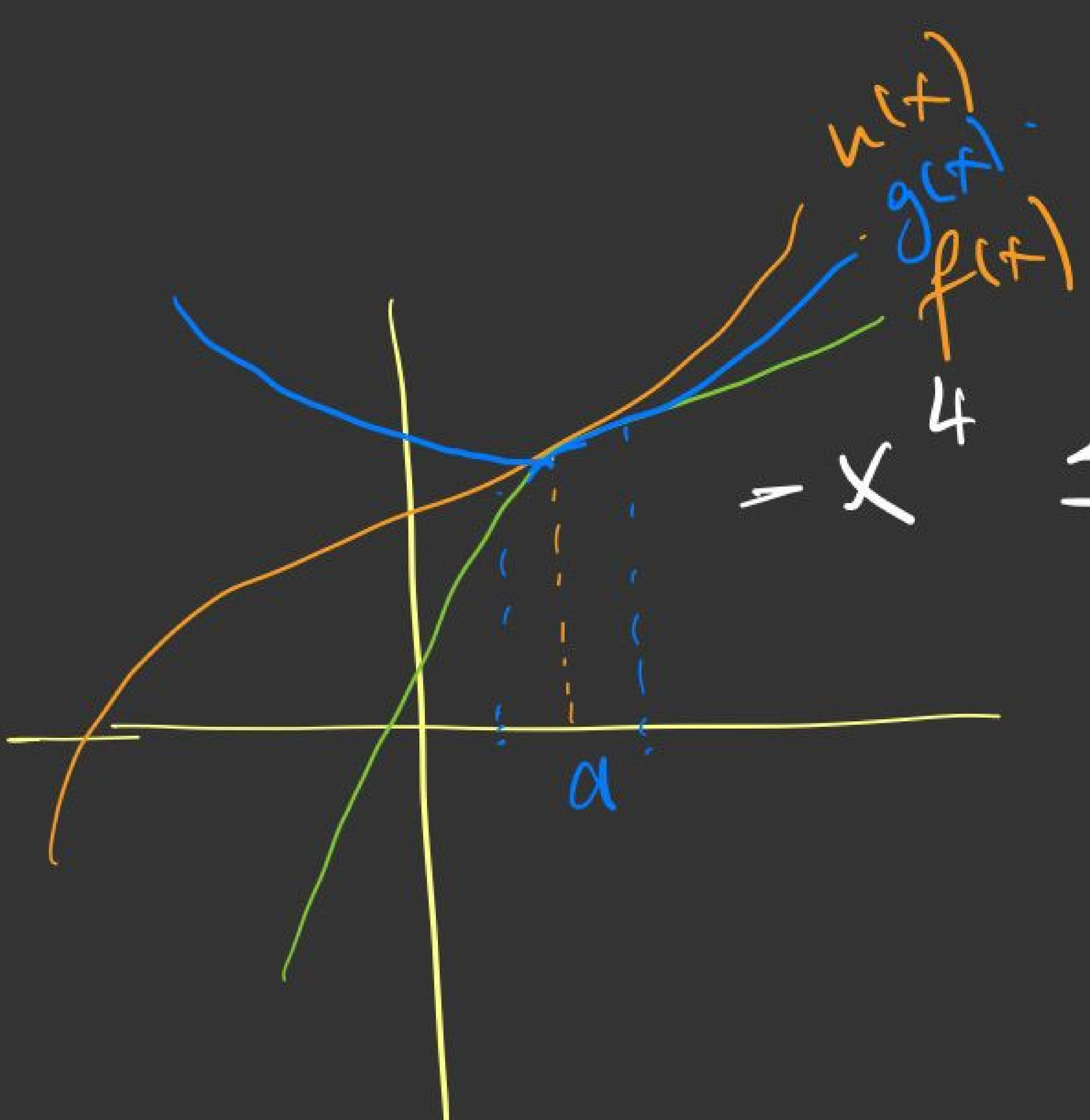
$(-a, a)$.

We know that $-1 \leq \cos x \leq 1$ $\forall x \in \mathbb{R}$.

" " " " " " $\forall x \in (-a, a)$.

$-1 \leq \cos(2/x) \leq 1$. $\forall x \in (-a, a)$.

↓ multiply the inequality with x^4 $(x^4 \geq 0)$.



$-x^4 \leq x^4 \cdot \cos(2/x) \leq x^4$

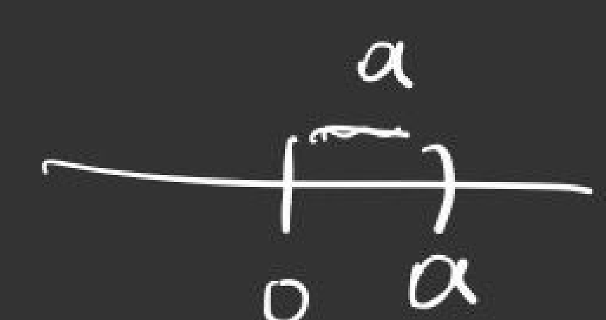
↓ Take limit as $x \to 0$ of each side.

$$\underbrace{\lim_{x\to 0} -x^4}_{\downarrow \; 0} \le \lim_{x\to 0} x^4\cdot\cos(2/x) \le \underbrace{\lim_{x\to 0} x^4}_{\downarrow \; 0}.$$

By Squeeze Thm, we must have $\lim_{x\to 0} x^4\cdot\cos(2/x)=0$.

$\square$.

(b) $\lim_{h\to 0^+} \sqrt{h}\cdot e^{\sin(\pi/h)} = 0.$



$(0,a)$

$-1\le \sin h \le 1 \quad \forall h\in\mathbb{R}.$

$-1\le \sin h \le 1 \quad \forall h\in(0,a).$

$-1\le \sin(\pi/h) \le 1 \quad \forall h\in(0,a).$

$\downarrow$ Take to the power of e to each side of inequality.

$\boxed{e\ge 1}$

$$\frac{1}{e} \le e^{\sin(\pi/h)} \le e. \quad \forall h\in(0,a).$$

$\downarrow$ multiply by $\sqrt{h}$

$(\sqrt{h}\ge 0).$

$$\frac{\sqrt{h}}{e} \le \sqrt{h}\cdot e^{\sin(\pi/h)} \le e\cdot\sqrt{h}.$$

Take lim as $h \to 0^+$ of each side

$$\lim_{h\to 0^+} \underbrace{\frac{\sqrt{h}}{e}}_{\downarrow \atop 0} \le \lim_{h\to 0^+} \sqrt{h}\cdot e^{\sin(\pi/h)} \le \lim_{h\to 0^+} \underbrace{e\cdot\sqrt{h}}_{\downarrow \atop 0}$$

By Squeeze Thm, $\lim_{h\to 0^+} \sqrt{h}\cdot e^{\sin(\pi/h)} = 0.$ $\square$.

#8: Evaluate $\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-2}$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2(1-2/x)}-2} = \lim_{x\to\infty} \frac{1}{x\sqrt{1-2/x}-2\cdot\frac{x}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\underbrace{x\Big(\underbrace{\sqrt{1-\underset{\downarrow \atop 0}{2/x}}}_{\downarrow \atop 1}-\underset{\downarrow \atop 0}{2/x}\Big)}_{\downarrow \atop 1}} = \underline{\underline{0.}}$$

#9: Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x^2}{x+1}-\dfrac{x^2}{x-1}\right)$

$(x-1)\qquad(x+1)$

$$= \lim_{x\to\infty}\frac{\left(x^2(x-1)-x^2(x+1)\right)}{(x-1)(x+1)}$$

$$= \lim_{x\to\infty}\frac{-2x^2}{x^2-1}$$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}(-2)}{\cancel{x^2}\underbrace{\left(1-\underbrace{1/x^2}_{\downarrow\,0}\right)}_{\downarrow\,1}} = -2.$$