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REC – II

#1: Sketch the graph of the following function and evaluate the limits

$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ (x-1)^2 & \text{if } x \geq 1 \end{cases}$$

a) $\lim\limits_{x \to -1} f(x) = ?$

$$\left[\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) \right]$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 2 - x = 2 - (-1) = 3.$$

$x < -1$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1.$$

$x > -1$

$$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x).$$

$\lim_{x \to -1} f(x)$ does not exist!
(DNE)

(b) $\lim_{x \to 0} f(x) = ?$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = \underline{\underline{0}}.$$

(c) $\lim_{x \to 1} f(x) = ?$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1.$$

x 1 $x < 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1)^2 = 0.$$

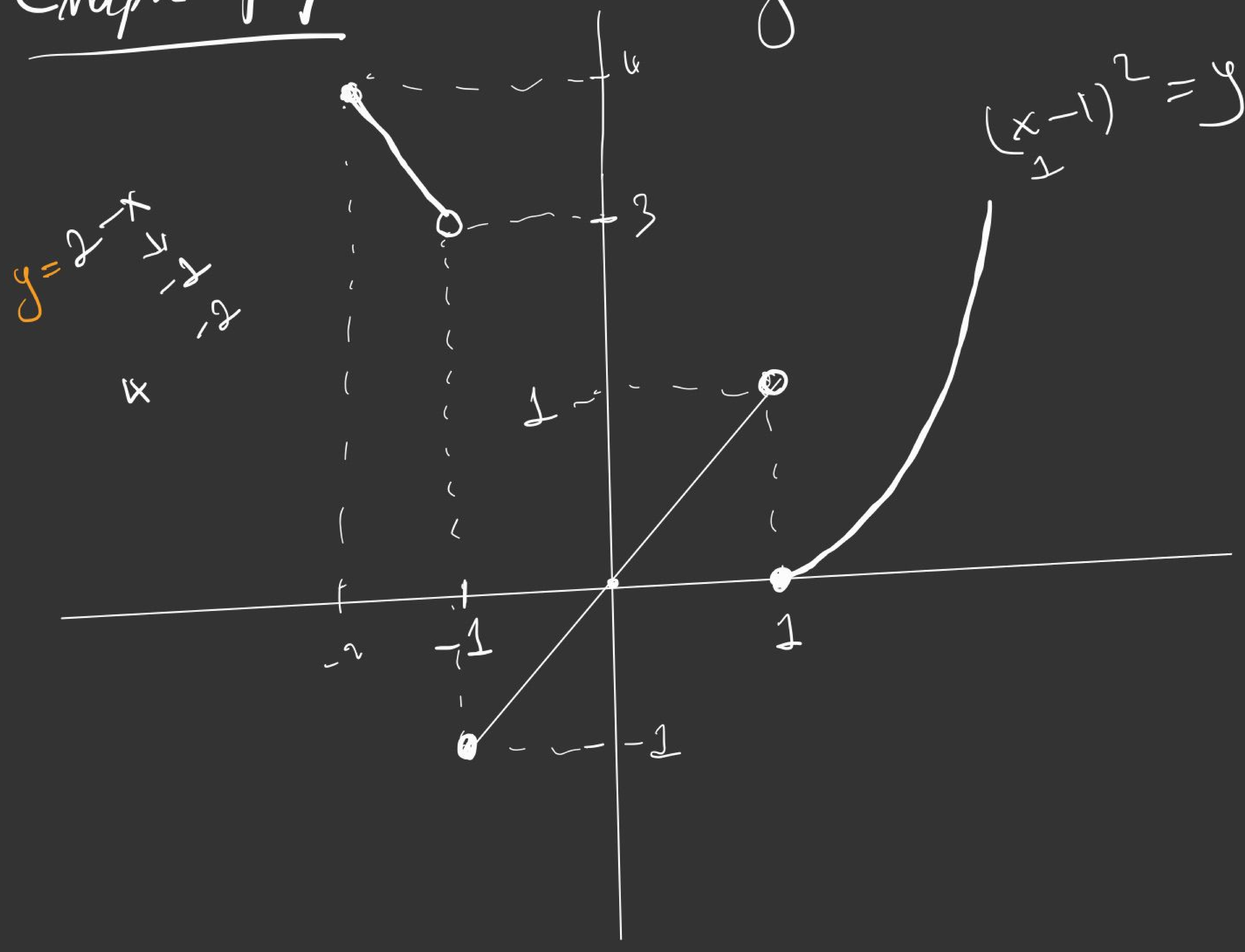
$x > 1$.
1 x

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1} f(x) \text{ DNE.}$$

Graph of f:

$y = f(x)$



#2: Let $f(x) = \frac{|x-2|}{x-2}$.

a) $\lim\limits_{x\to 0} f(x) = ?$

$$= \lim_{x\to 0} \frac{|x-2|}{x-2} = \frac{|-2|}{-2} = -1.$$

(b) $\lim\limits_{x\to 2} f(x) = ?$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{-x+2}{x-2} = -1.$$

$x < 2$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1.$$

$x > 2$.

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

Therefore, $\lim\limits_{x\to 2} f(x)$ DNE.

#3: Evaluate the limit if it exists

a) $\lim_{x \to 1} \frac{x^3-1}{x^2-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}.(x^2+x+1)}{\cancel{(x-1)}(x+1)}$

$= \frac{3}{2}$

b) $\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h} =$

$= \lim_{h \to 0} \frac{(\sqrt{1+h}-1)}{h} \cdot \frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}$

$= \lim_{h \to 0} \frac{(\sqrt{1+h})^2-1}{h.(\sqrt{1+h}+1)}$

$= \lim_{h \to 0} \frac{\cancel{1}+h-\cancel{1}}{\cancel{h}.(\sqrt{1+\underset{0}{h}}+1)} \quad \cancel{h} \quad = \frac{1}{2}$

c) $\lim_{x \to 1} \dfrac{\sqrt{x} - x^2}{1 - \sqrt{x}} = \lim_{x \to 1} \dfrac{\sqrt{x} - (\sqrt{x})^4}{1 - \sqrt{x}}$.

$= \lim_{x \to 1} \dfrac{\sqrt{x} \cdot \left(1 - (\sqrt{x})^3\right)}{1 - \sqrt{x}}$

$1 - t^3 = (1-t) \cdot (1 + t + t^2)$

$t = \sqrt{x}$

$= \lim_{x \to 1} \dfrac{\sqrt{x} \cdot \cancel{(1 - \sqrt{x})}(1 + \sqrt{x} + x)}{\cancel{(1 - \sqrt{x})}}$

$= \underline{\underline{3}}$.

#4: Let $f(x) = x - \lfloor x \rfloor$

[$g(x) = \lfloor x \rfloor \rightarrow$ the greatest integer function

It gives the largest integer less than or equal to x.]

(a) If n is an integer, evaluate

$\boxed{n \in \mathbb{Z}}$.

(i) $\lim_{x \to n^-} f(x)$

(ii) $\lim_{x \to n^+} f(x)$.

(i) $\lim_{x \to n^-} f(x) = \lim_{x \to n^-} \frac{x}{n} - \frac{\lfloor x \rfloor}{n-1} = n-(n-1)$

$= 1.$

$x < n$

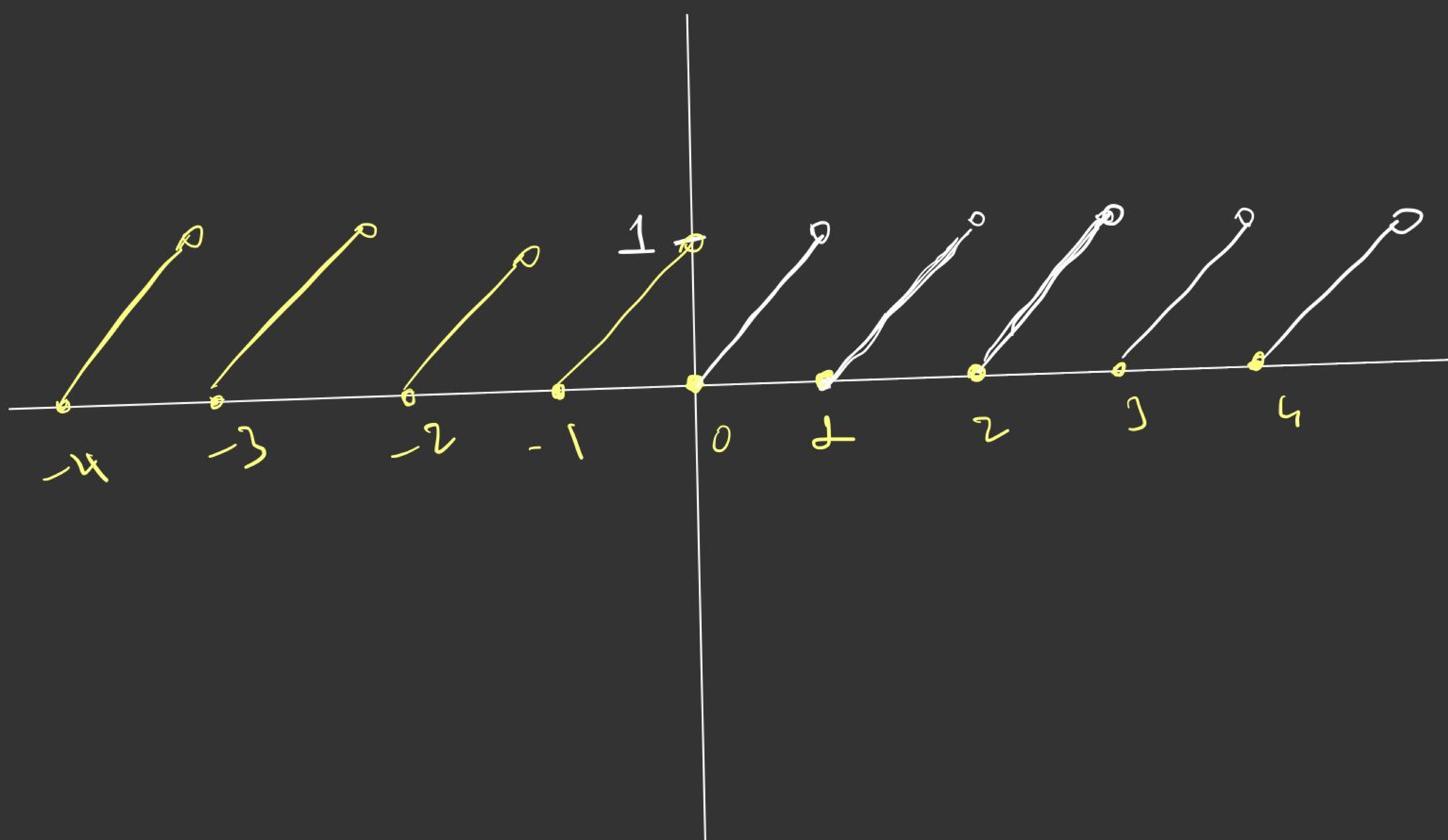
n

(ii) $\lim_{x \to n^+} f(x) = \lim_{x \to n^+} \frac{x}{n} - \frac{\lfloor x \rfloor}{n} = 0.$

$n \quad x \quad x > n.$

(b) Sketch the graph of f.

$f(n) = n - \frac{\lfloor n \rfloor}{n} = n - n = 0. \quad \forall n \in \mathbb{Z}.$



Take $0 \leq x < 1$.

$y = f(x) = x - \lfloor x \rfloor$

$y = x - 0$

$y = x$.

Take $1 \leq x < 2$.

$y = f(x) = x - \lfloor x \rfloor$

$y = x - 1$.

Take $n \leq x < n+1$

$y = f(x) = x - \lfloor x \rfloor$

$= x - n$.

(c) For what values of a, does $\lim_{x \to a} f(x)$ exist?

For integer values of n, from part (a) we know that $\lim_{x \to n^-} f(x) = 1 \neq \lim_{x \to n^+} f(x) = 0$.

Therefore $\lim_{x \to n} f(x)$ DNE.

Take $a \in \mathbb{R} \setminus \mathbb{Z}$. Then $\lim_{x \to a} f(x)$ exists

since $\lim_{x \to a^-} f(x) = \underline{a - \lfloor a \rfloor} = \lim_{x \to a^+} f(x)$.

Therefore, we must have $\boxed{a \in \mathbb{R} \setminus \mathbb{Z}}$.

#5: If $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$, show that

$\lim\limits_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \underbrace{\lfloor x \rfloor}_{1} + \underbrace{\lfloor -x \rfloor}_{-2} = -1$$

$x \quad 2$ $\qquad \boxed{x<2} \Rightarrow \boxed{-x>-2}$

$-2 \quad -x$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \underbrace{\lfloor x \rfloor}_{2} + \underbrace{\lfloor -x \rfloor}_{-3} = -1$$

$2 \quad x$ $\qquad \boxed{x>2} \Rightarrow \boxed{-x<-2}$

$-3 \quad -x \quad -2$

$$\boxed{\lim_{x\to 2} f(x) = -1} \qquad \neq$$

$$f(2) = \underbrace{\lfloor 2 \rfloor}_{=2} + \underbrace{\lfloor -2 \rfloor}_{=-2} = 0$$

#6: Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$ $\quad 3-x = y$

Change the variable

$y = 3-x.$

$\downarrow \quad \downarrow$

$1 \quad 2$

$$= \lim_{y\to 1} \frac{\sqrt{y+3}-2}{\sqrt{y}-1}$$

$$= \lim_{y\to 1} \frac{\sqrt{y+3}-2}{\sqrt{y}-1} \cdot \frac{\sqrt{y+3}+2}{\sqrt{y+3}+2}$$

$$= \lim_{y\to 1} \frac{y+3-4}{(\sqrt{y}-1)(\sqrt{y+3}+2)} \qquad y-1 = (\sqrt{y}-1)\cdot(\sqrt{y}+1)$$

$$= \lim_{y\to 1} \frac{\cancel{(\sqrt{y}-1)}(\sqrt{y}+1)}{\cancel{(\sqrt{y}-1)}(\sqrt{y+3}+2)}$$

$$= \frac{2}{4} = \underline{\underline{\frac{1}{2}}}.$$

#7: Prove that

(a) $\lim_{x\to 0} x^4 \cdot \cos(2/x) = 0.$

Recall: Squeeze Thm (Sandwich Thm)

Suppose that $f(x) \le g(x) \le h(x)$ $\forall x \in I$ where I is an open interval containing a, and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = L$.

We know that $-1 \le \cos x \le 1$ $\forall x \in \mathbb{R}$.

$\forall x \in (-a, a)$ $\quad -a \quad 0 \quad a$

It is also true that $-1 \le \cos(2/x) \le 1$

$\forall x \in (-a, a)$

$\downarrow$ multiply it with x^4 $(x^4 \ge 0)$.

$$-x^4 \le x^4 \cdot \cos(2/x) \le x^4 \quad \forall x \in (-a, a).$$

$\downarrow$ Take lim as $x \to 0$ of each side

$$\underbrace{\lim_{x\to 0} -x^4}_{\downarrow\ 0} \le \underbrace{\lim_{x\to 0} x^4\cdot\cos(2/x)} \le \underbrace{\lim_{x\to 0} x^4}_{\downarrow\ 0.}$$

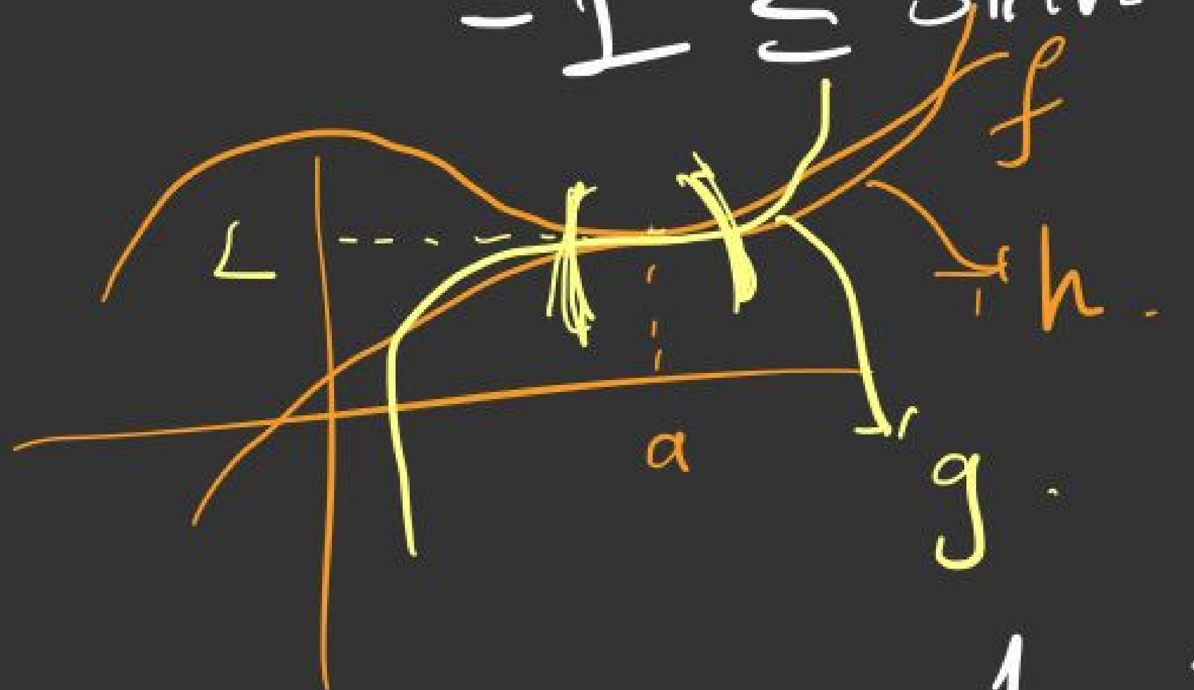
By Squeeze Theorem, we must have $\lim_{x\to 0} x^4\cdot\cos(2/x) = 0$. $\square$.

(b) $\underline{\lim_{h\to 0^+}} \sqrt{h}\cdot e^{\sin(\pi/h)} = 0$.

$-1 \le \sin h \le 1$ $\forall h\in\mathbb{R}$.

$\forall h\in(0,a)$.

$(0,a)$



$-1 \le \sin(\pi/h) \le 1$ $\forall h\in(0,a)$.

$\downarrow$ Take to the power e of each func. in the inequality $e>1$.

$-e \le e^{\sin(\pi/h)} \le e$.

$\downarrow$ multiply by $\sqrt{h}$ $(\sqrt{h}\ge 0)$.

$-\sqrt{h}\cdot e \le \sqrt{h}\cdot e^{\sin(\pi/h)} \le e\cdot\sqrt{h}$.

$\downarrow$ Take lim as $h\to 0^+$ of each side.

$$\lim_{h\to 0^+} -\sqrt{h}\cdot e \le \lim_{h\to 0^+} \sqrt{h}\cdot e^{\sin(\pi/h)} \le \lim_{h\to 0^+} \sqrt{h}\cdot e$$

$$\underbrace{\lim_{h\to 0^+} -\sqrt{h}\cdot e}_{\downarrow\ 0} \qquad \underbrace{\lim_{h\to 0^+} \sqrt{h}\cdot e}_{\downarrow\ 0.}$$

By Squeeze Thm, we have

$$\lim_{h\to 0^+} \sqrt{h}\cdot \sin(\pi/h) = 0.$$

$\square$.

#8: Evaluate $\lim_{x\to\infty} \dfrac{1}{\sqrt{x^2-2x}-2}$.

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2(1-2/x)}-2}$$

$$= \lim_{x\to\infty} \frac{1}{x\sqrt{1-2/x}-2\cdot x\cdot \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\underset{\downarrow\ \infty}{x}\underbrace{\left(\sqrt{1-\underset{\downarrow\ 0}{2/x}}-\underset{\downarrow\ 0}{(2/x)}\right)}_{\downarrow\ 1}} = 0.$$

#9: Evaluate $\lim_{x \to \infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1} \right)$

$(x-1) \quad (x+1)$

$$= \lim_{x \to \infty} \frac{x^2(x-1) - x^2(x+1)}{(x-1)(x+1)}$$

$$= \lim_{x \to \infty} \frac{-2x^2}{x^2 - 1}$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2}(-2)}{\underbrace{\cancel{x^2}(1 - \underbrace{1/x^2}_{\to 0})}_{\to 1}} = -2.$$