REC – III

#1: Find m so that $g(x) = \begin{cases} x-m & \text{if } x < 3 \\ 1-mx & \text{if } x \geq 3 \end{cases}$

is continuous for all x.

[f is cont. at $x=c$ $\iff$ $\lim_{x\to c} f(x) = f(c)$]

Observe that, since $x-m$ is a polynomial function, it is continuous for $x<3$. Similarly, $1-mx$ is continuous for $x>3$.

we only need to check the continuity for the point $x=3$:

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} x-m = 3-m.$$

$x<3$.

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} 1-m.x = 1-3m.$$

$x>3$.

$$f(3) = 1-3m.$$

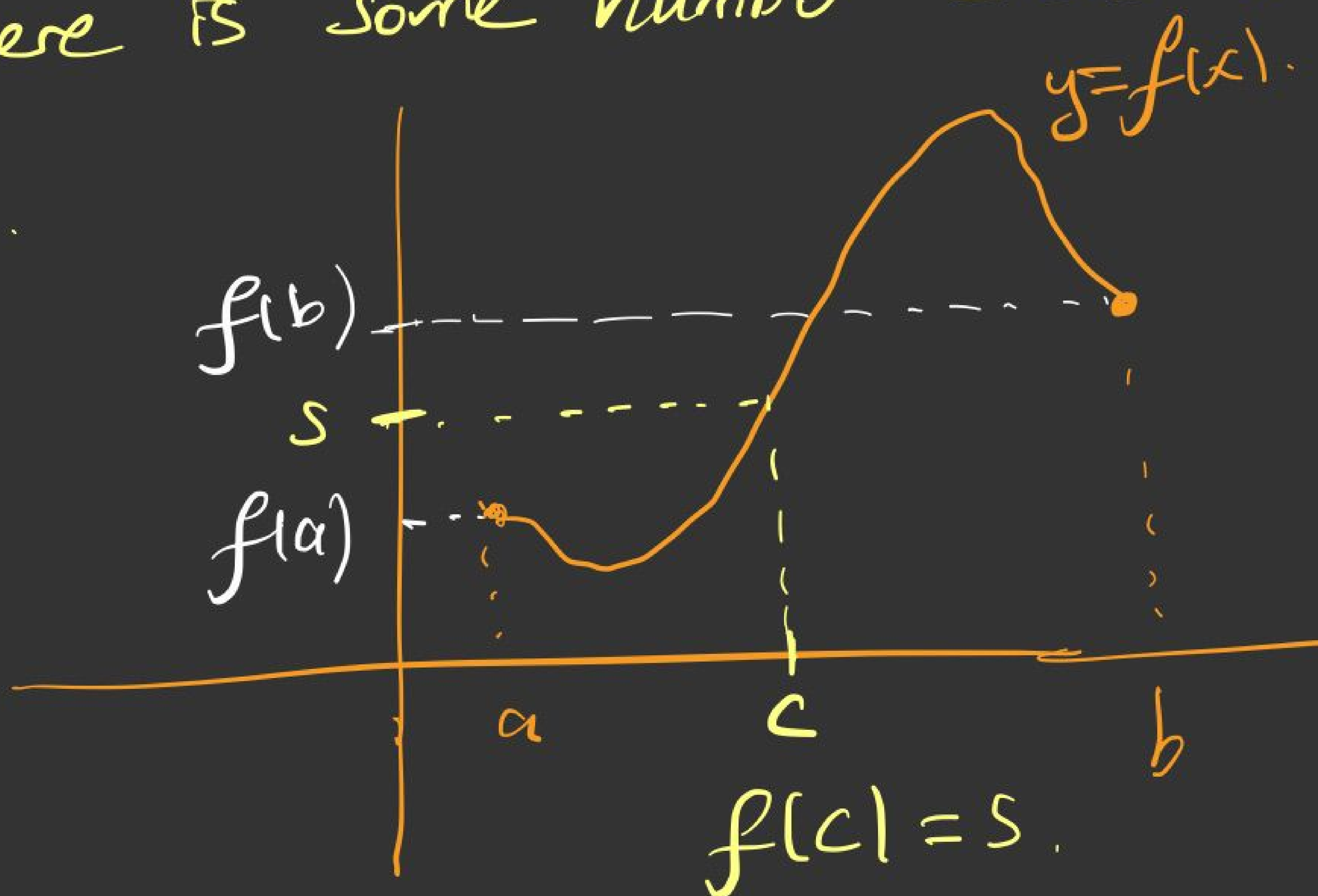
They must be equal.

$$1-3m = 3-m \Rightarrow -2m = 2 \Rightarrow \boxed{m=-1}$$

#2: Show that there is some a with $0 < a < 2$ such that $a^2 + \cos(\pi a) = 4$.

Recall: Intermediate Value Theorem

If f is a continuous function on some closed interval $[a,b]$ and if s is a value between $f(a)$ & $f(b)$ then there is some number $c \in [a,b]$ such that $f(c) = s$.



Let $f(x) = x^2 + \cos(\pi x)$

Observe that f is a continuous function on $[0,2]$. (why? since f is a linear combination of a polynomial function & cosine function, which are both continuous functions.)

$$f(0) = 0^2 + \underbrace{\cos(\pi \cdot 0)}_{1} = 1$$

$$\wedge$$
$$4$$
$$\wedge$$

$$f(2) = 2^2 + \underbrace{\cos(2\pi)}_{1} = 5$$

Since f is cont. on $[0,2]$ and since

$\underset{f(0)}{\overset{\shortparallel}{1}} < 4 < \underset{f(2)}{\overset{\shortparallel}{5}}$ by IVT, there is some number

$a \in [0,2]$ s.t $f(a) = a^2 + \cos(\pi . a) = 4$.

$1 = f(0) \neq 4 \Rightarrow a \neq 0$

$5 = f(2) \neq 4 \Rightarrow a \neq 2$.

Therefore, we must have $a \in (0,2)$.

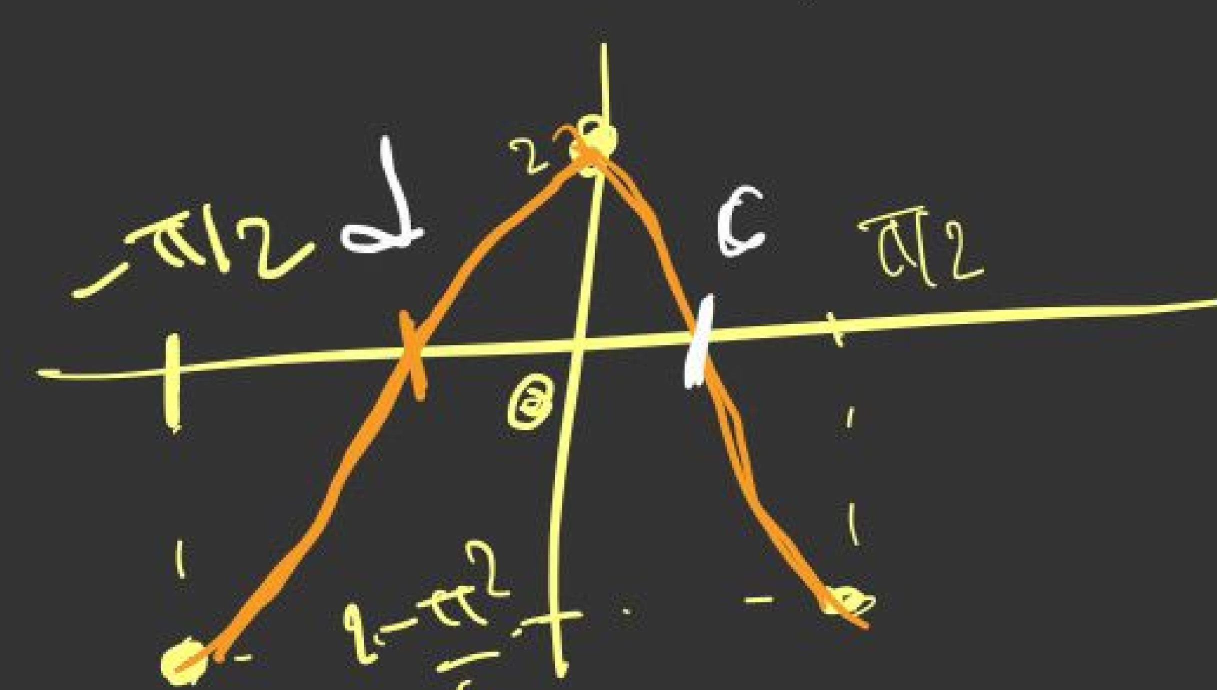
#3: Show that the equation $\cos x = x^2 - 1$ has at least two solutions.

$\cos x - x^2 + 1 = 0$.

Let $f(x) = \cos x - x^2 + 1$. Then observe that f is cont. everywhere (since $-x^2+1$ & $\cos x$ are cont. everywhere)

$$f(\underset{=}{0}) = \underbrace{\cos 0}_{1} - 0^2 + 1 = 2 > 0$$

$$f(\underset{=}{\tfrac{\pi}{2}}) = \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} - \frac{\pi^2}{4} + 1 = 1 - \frac{\pi^2}{4} < 0$$



- Since f is cont. on $[0, \pi/2]$ and

$$\underbrace{1-\frac{\pi^2}{4}}_{=f(\pi/2)} < 0 < \underset{f(0)}{\overset{\shortparallel}{2}}$$

, by IVT, there is some

$c \in [0, \pi/2]$ s.t $f(c) = \cos(c) - c^2 + 1 = 0$.

So c is a solution of the eqn.

$$f(-\pi/2) = \underbrace{\cos\left(-\frac{\pi}{2}\right)}_{=0} - \frac{\pi^2}{4} + 1 = 1 - \frac{\pi^2}{4} < 0.$$

- Since f is cont. on $[-\pi/2, 0]$ &

$$\underbrace{1-\frac{\pi^2}{4}}_{=f(-\pi/2)} < 0 < \underset{f(0)}{\overset{\shortparallel}{2}}$$

, by IVT, there is some

$d \in [-\pi/2, 0)$ s.t $f(d) = \cos d - d^2 + 1 = 0$.

Therefore, d is another solution.

Hence, $\cos x - x^2 + 1 = 0$ has at least two solutions.

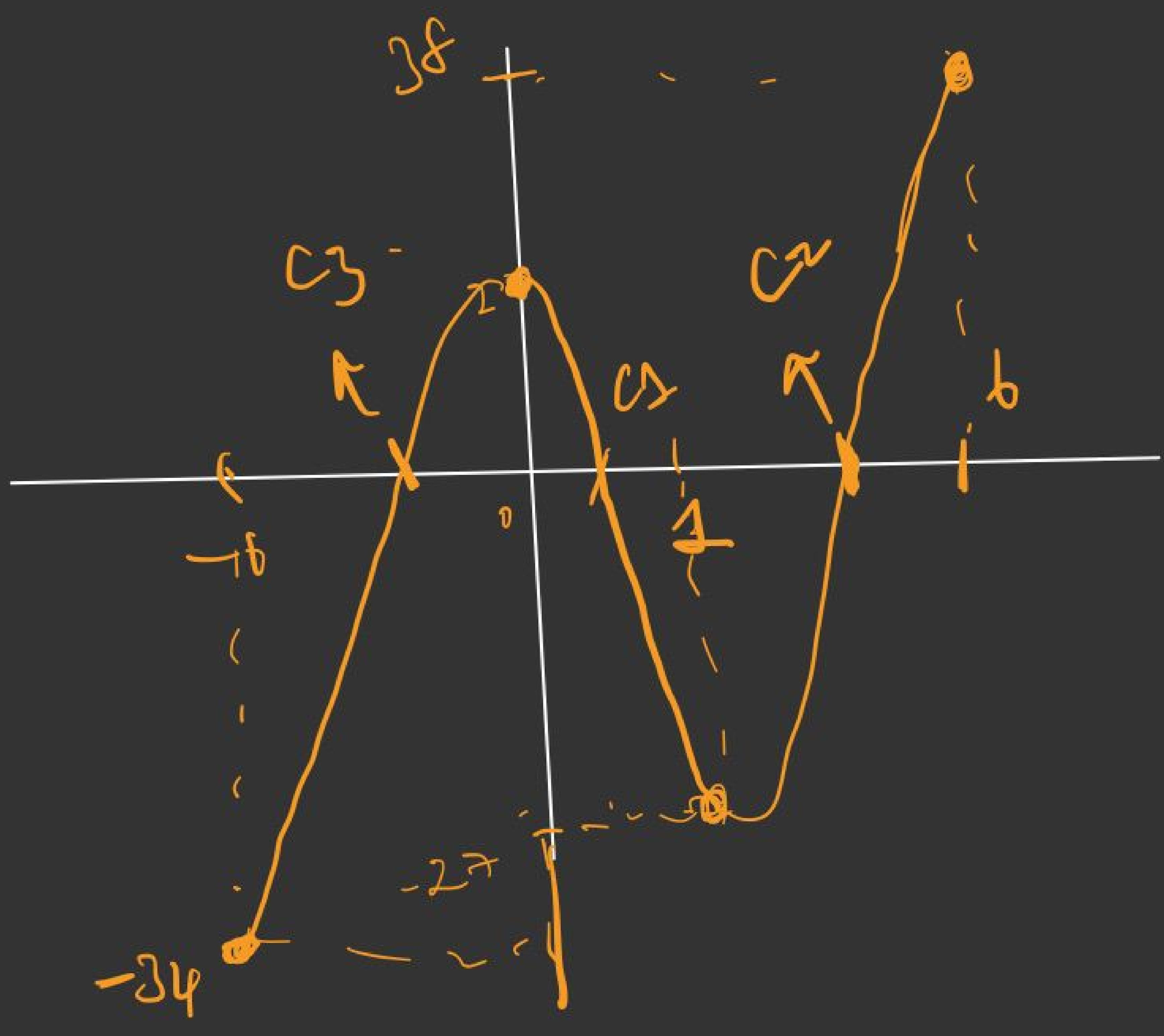
#4: Show that the equation $x^3 - 30x + 2 = 0$ has at least three solutions

Let $f(x) = x^3 - 30x + 2$. Then since f is a polynomial function, it is continuous everywhere.

$$f(0) = 0^3 - 30 \cdot 0 + 2 = 2 > 0$$
$$f(1) = 1 - 30 + 2 = -27 < 0$$

Since f is cont. on $[0, 1]$ and since $\underbrace{-27}_{=f(1)} < 0 < \underbrace{2}_{=f(0)}$, by IVT, there is some $c_1 \in [0, 1]$ s.t $f(c_1) = c_1^3 - 30c_1 + 2 = 0$.

So, c_1 is a solution of the equation.



$$f(6) = 216 - 180 + 2 = 38 > 0.$$

- Since f is cont on $[1,6]$ &

$\underset{f(1)}{\underset{\|}{-27}} < 0 < \underset{f(6)}{\underset{\|}{38}}$, by IVT there is some

$c_2 \in [1,6]$ s.t $f(c_2) = 0$.

So c_2 is also a solution.

$$f(-6) = -216 + 180 + 2 = -34 < 0.$$

- Since f is cont on $[-6,0]$ and

since $\underbrace{-34}_{=f(-6)} < 0 < \underbrace{2}_{=f(0)}$, by IVT, there

is some $c_3 \in [-6, 0]$ s.t $f(c_3) = 0$.

So c_3 is another solution.

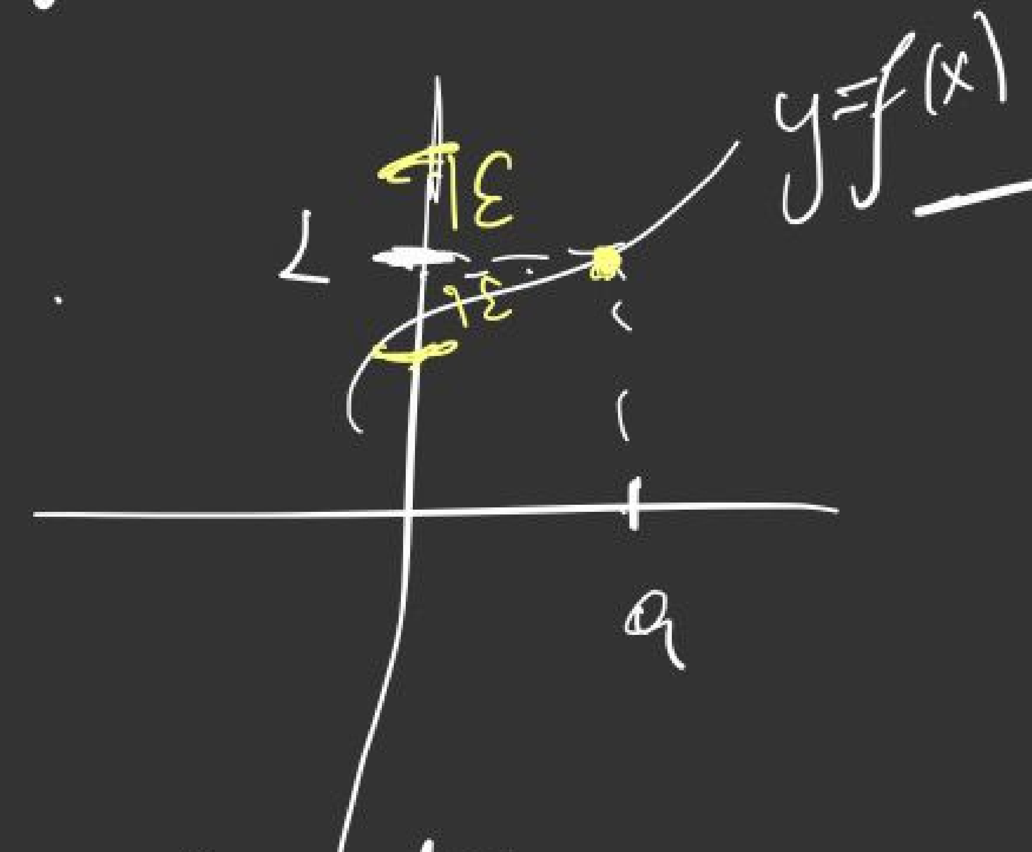
Hence, $f(x) = x^3 - 30x + 2$ has at least

three solutions.

#5: Use the formal definition of the limit to verify the following.

(a) $\lim_{x \to c} \underbrace{(ax+b)}_{f(x)} = \overset{L}{a \cdot c + b}$ for any $a, b, c \in \mathbb{R}$.

$$\left[\lim_{x \to a} f(x) = L \iff \underset{\text{For all}}{\forall} \varepsilon > 0 \ \underset{\text{there exists}}{\exists} \delta > 0 \text{ such that if } 0 < |x-a| < \delta, \text{ we have } |f(x) - L| < \varepsilon. \right]$$



Let $\varepsilon > 0$ be given. Then there exists

$\boxed{\delta = \frac{\varepsilon}{|a|}} > 0$ such that if $0 < |x-c| < \delta$

we have $|f(x) - L| = |ax + b - (ac+b)|$

$= |ax + \cancel{b} - ac - \cancel{b}| = |ax - ac|$

$= |a(x-c)| = |a| \cdot \underbrace{|x-c|}_{< \delta} < |a| \cdot \overset{\text{choose}}{\delta} = \varepsilon$

$$\boxed{\delta = \frac{\varepsilon}{|a|}.}$$

(b) $\lim_{x \to 2} \frac{x-2}{1+x^2} = 0.$

Let $\varepsilon > 0$ be given. Then there exists $\boxed{\delta = \varepsilon} > 0$ s.t if $0 < |x-2| < \delta$

we have $|f(x) - L| = \left| \frac{x-2}{1+x^2} - 0 \right|$

$$= \left| \frac{x-2}{1+x^2} \right| = \frac{|x-2|}{1+x^2} = |x-2| \cdot \frac{1}{1+x^2}$$

$\leq 1.$

$1+x^2 \geq 1.$

$\frac{1}{1+x^2} \leq 1.$

$$\leq |x-2| < \delta = \varepsilon$$

choose

$$\boxed{\delta = \varepsilon}$$

(c) $\lim_{x \to 3} \sqrt{2x+3} = 3$.

Let $\varepsilon > 0$ be given. Then there exists some $\boxed{\delta = \frac{\varepsilon}{2}} > 0$ such that

if $0 < |x-3| < \delta$ we have

$$(\sqrt{2x+3})^2 - 3^2$$

$$|f(x) - L| = \left|\sqrt{2x+3} - 3\right| = \frac{|(2x+3) - 9|}{|\sqrt{2x+3} + 3|}$$

$$= \frac{|2x-6|}{|\sqrt{2x+3}+3|} = \frac{2|x-3|}{|\underbrace{\underbrace{\sqrt{2x+3}}_{\geq 0} + 3}_{\geq 3 > 1}|}$$

$$\left(\frac{1}{|\sqrt{2x+3}+3|} < 1 \right)$$

$$= 2|x-3| \cdot \underbrace{\frac{1}{|\sqrt{2x+3}+3|}}_{< 1} < 2\underbrace{|x-3|}_{< \delta} < 2\delta$$

$\overset{\shortparallel}{\varepsilon}$

choose $\boxed{\delta = \varepsilon/2}$