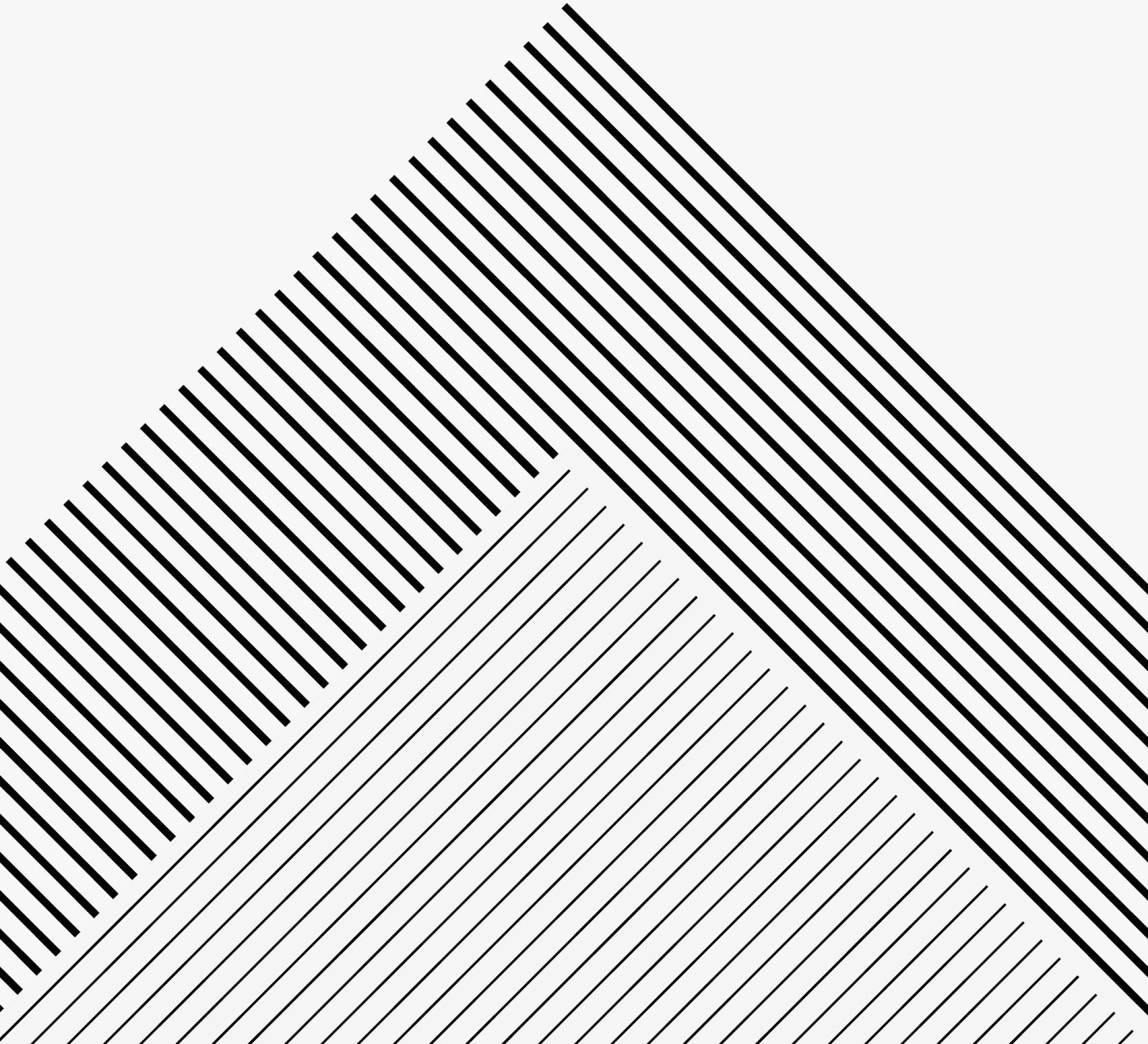


REC – III

#1: Find m so that $g(x) = \begin{cases} x-m & \text{if } x<3 \\ 1-mx & \text{if } x \geq 3. \end{cases}$

is continuous for all x.

$$\left[\lim_{x \to c} f(x) = f(c) \iff f \text{ is cont. at } x=c. \right]$$

observe that $x-m$ is a continuous function (since it is a polynomial) everywhere. $\Rightarrow$ therefore it is also continuous for $x<3$.

Similarly, $1-mx$ is also continuous (being a polynomial) for $x>3$.

Therefore, we only need to check the continuity at the point $\underline{x=3}$:

$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} x-m = \underline{3-m}.$$

$x<3$.

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} 1-mx = \underline{1-3m}.$$

$x>3$.

$$g(3) = \underline{1-3m}$$

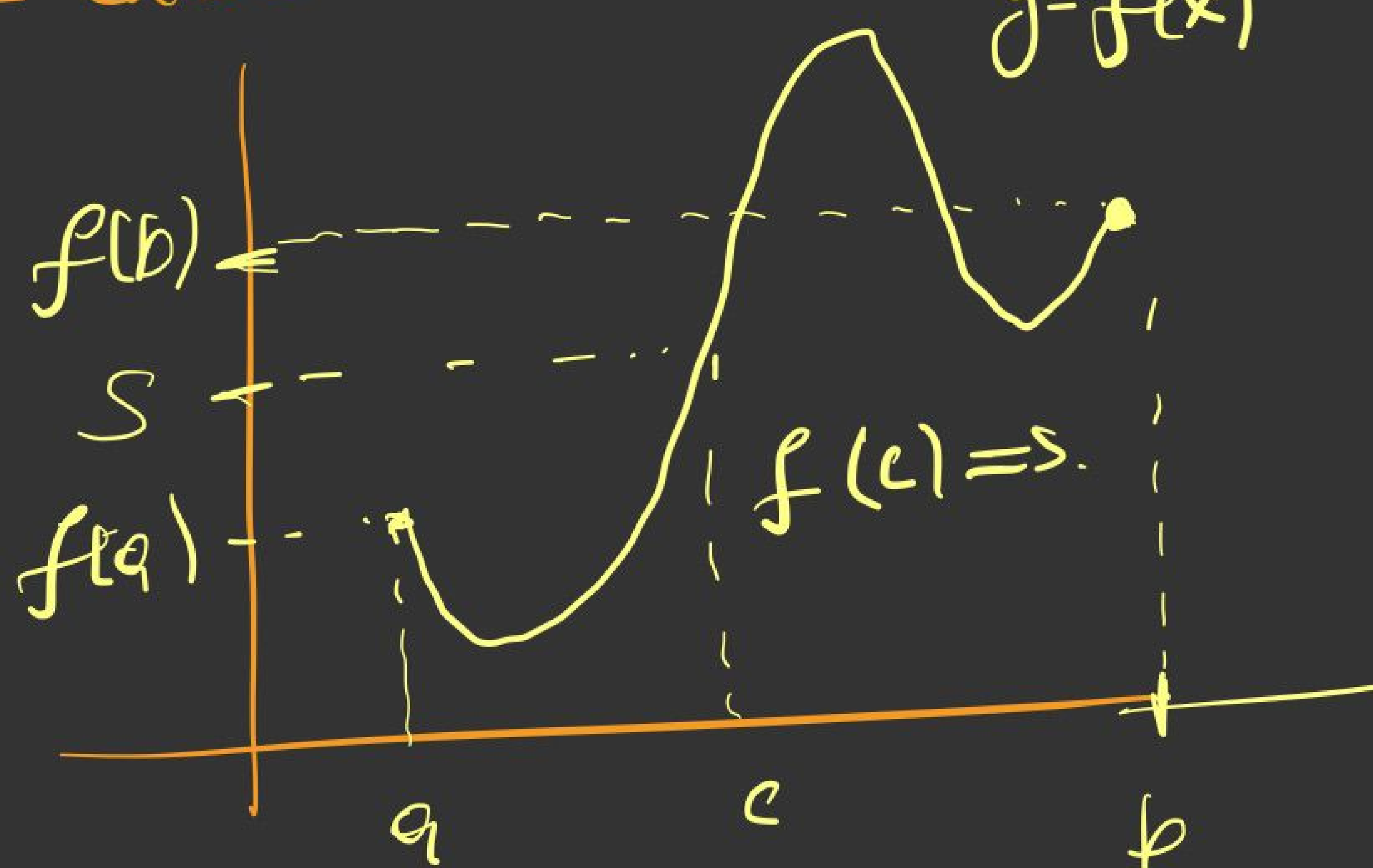
They must be equal.

$$3-m = 1-3m \Rightarrow -2m = 2 \Rightarrow \boxed{m=-1}$$

#2: Show that there is some a with $0 < a < 2$ such that $\underbrace{a^2 + \cos(\pi a)}_{f(a)} = 4$. $\boxed{f(a) = 4}$

Recall: Intermediate Value Theorem

If f is a continuous function on some closed interval $[a, b]$ and if s is a value between $f(a)$ & $f(b)$ then there exists some number $c \in [a,b]$ such that $f(c) = s$.



Let $f(x) = x^2 + \cos(\pi x)$.

Since x^2 is cont. everywhere & $\cos(\pi x)$ is also cont. everywhere, f will also be cont. on everywhere.

Therefore f is cont. on $[0,2]$.

$$f(0) = 0^2 + \underbrace{\cos(\pi \cdot 0)}_{1} = 1.$$

$$\wedge$$
$$4$$
$$\wedge$$

$$f(2) = 2^2 + \underbrace{\cos(2\pi)}_{1} = 5.$$

Since $\underset{f(0)}{\overset{\shortparallel}{1}} < 4 < \underset{f(2)}{\overset{\shortparallel}{5}}$, by IVT

there exists some $a \in [0,2]$ s.t $\underset{a^2+\cos(\pi a)}{\overset{\shortparallel}{f(a)}} = 4$.

$1 = f(0) \neq 4$ so $a \neq 0$.

$5 = f(2) \neq 4$ so $a \neq 2$.

Therefore, there is some $a \in (0,2)$ s.t $f(a)=4$.

#3: Show that the equation $\cos x = x^2 - 1$ has at least two solutions.

$\underset{S}{\overset{\shortparallel}{\cos x - x^2 + 1 = 0}}$

Let $f(x) = \cos x - x^2 + 1$. Then since cosine func. is cont. everywhere and $-x^2+1$ is cont everywhere (being a polynomial) then f is also cont. everywhere.

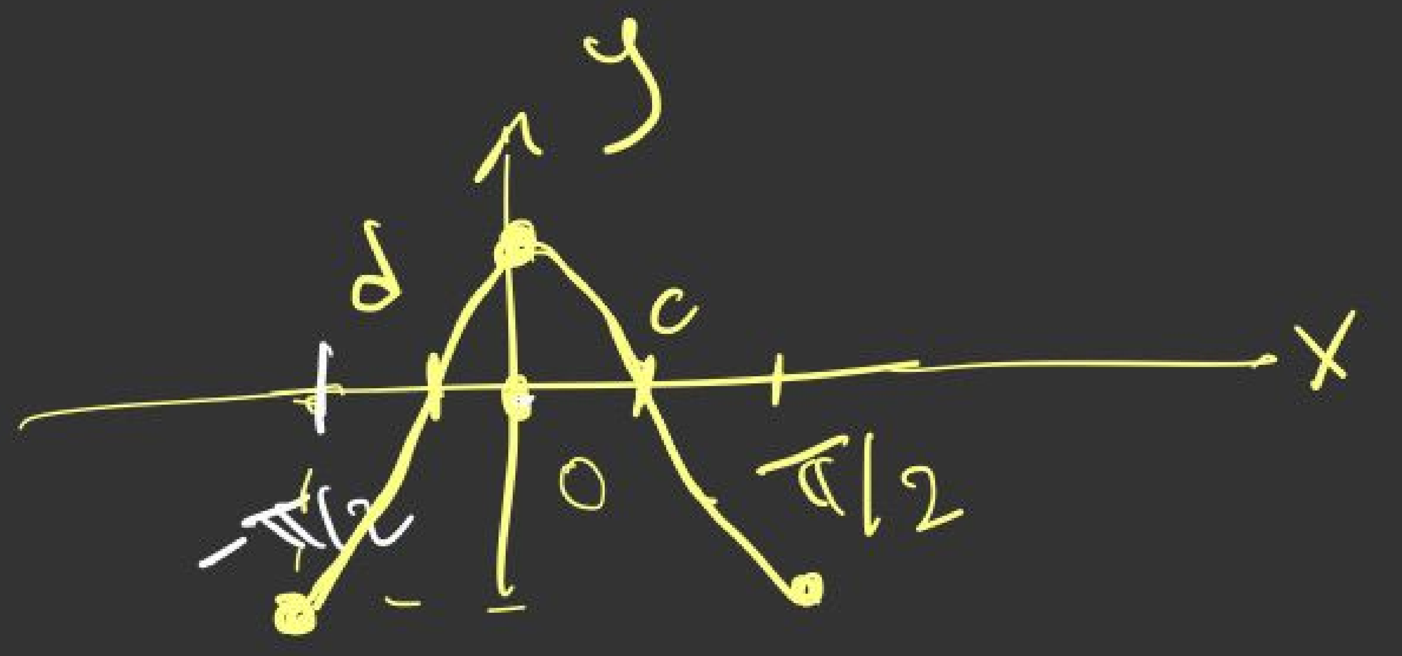
$f(0) = \underbrace{\cos 0}_{1} - 0^2 + 1 = 2 > 0$

$f(\pi/2) = \underbrace{\cos(\pi/2)}_{0} - \frac{\pi^2}{4} + 1 = 1 - \frac{\pi^2}{4} < 0$.

Since $\underset{f(\pi/2)}{\overset{\shortparallel}{1-\frac{\pi^2}{4}}} < 0 < \underset{f(0)}{\overset{\shortparallel}{2}}$, and since f is cont. on $[0, \pi/2]$ so by IVT, there is some $c \in [0, \pi/2]$ such that

$f(c) = 0$ So c is a solution of that equation.

$\cos(c) - c^2 + 1 = 0$.



$$f(-\pi/2) = \underbrace{\cos(-\pi/2)}_{0} - \frac{\pi^2}{4} + 1$$

$$= 1 - \frac{\pi^2}{4} < 0.$$

Since $\underbrace{1 - \frac{\pi^2}{4}}_{\substack{\shortparallel \\ f(-\pi/2)}} < 0 < \underset{\substack{\shortparallel \\ f(0)}}{2}$ and since f is

cont. on $[-\pi/2, 0]$, by IVT, there is

some $d \in [-\pi/2, 0]$ s.t. $f(d) = \cos d - d^2 + 1 = 0$.

So d is another solution of the equ.

Since c & d are both solutions of f,

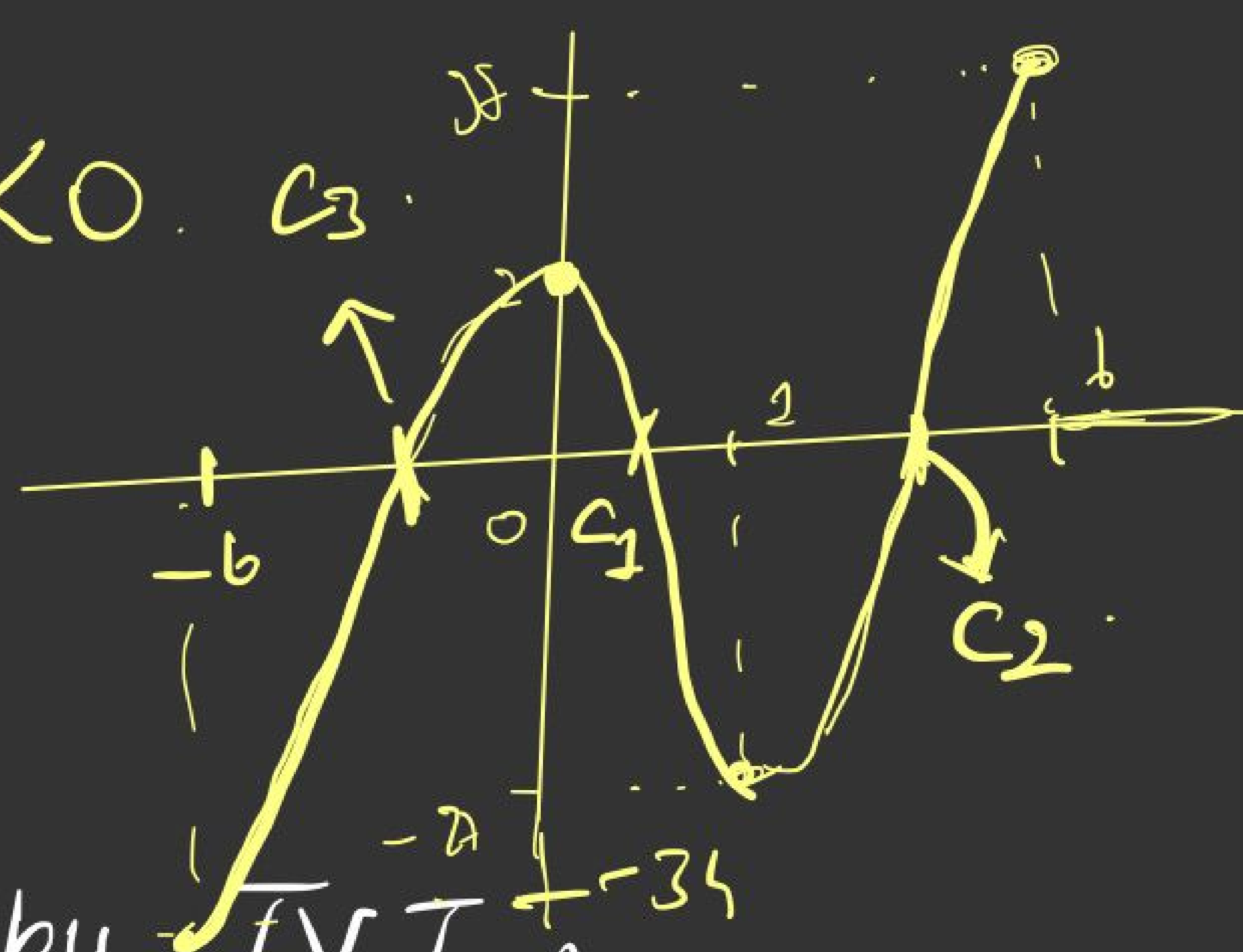
f must have at least two solutions $\square$.

#4: Show that the equation $x^3 - 30x + 2 = 0$ has at least three solutions.

Let $f(x) = x^3 - 30x + 2$. Since f is a polynomial function f is continuous everywhere.

$f(0) = 2 > 0$.

$f(1) = 1 - 30 + 2 = -27 < 0$.



- Since $\underset{f(1)}{\underset{\|}{-27}} < 0 < \underset{f(0)}{\underset{\|}{2}}$ and

since f is cont. on $[0, 1]$, by IVT, there is some $c_1 \in [0,1]$ s.t $f(c_1) = 0$.

So c_1 is a solution.

$f(6) = 216 - 180 + 2 = 38 > 0$.

- Since $\underset{f(1)}{\underset{\|}{-27}} < 0 < \underset{f(6)}{\underset{\|}{38}}$ and since f is cont on $[1, 6]$ by IVT, there is some $c_2 \in [1, 6]$ s.t $f(c_2) = 0$.

So c_2 is another solution.

$$f(-6) = -216 + 180 + 2 = -34 < 0.$$

• Since $\underset{\substack{\shortparallel \\ f(-6)}}{-34} < 0 < \underset{\substack{\shortparallel \\ f(0)}}{2}$ and since f is cont

on $[-6, 0]$, by IVT, there is some

number $c_3 \in [-6, 0]$ s.t. $f(c_3) = 0$.

So c_3 is also a solution.

$\{c_1, c_2, c_3\}$.

So $x^2 - 30x + 2 = 0$ has at least 3 solutions

$\square$.

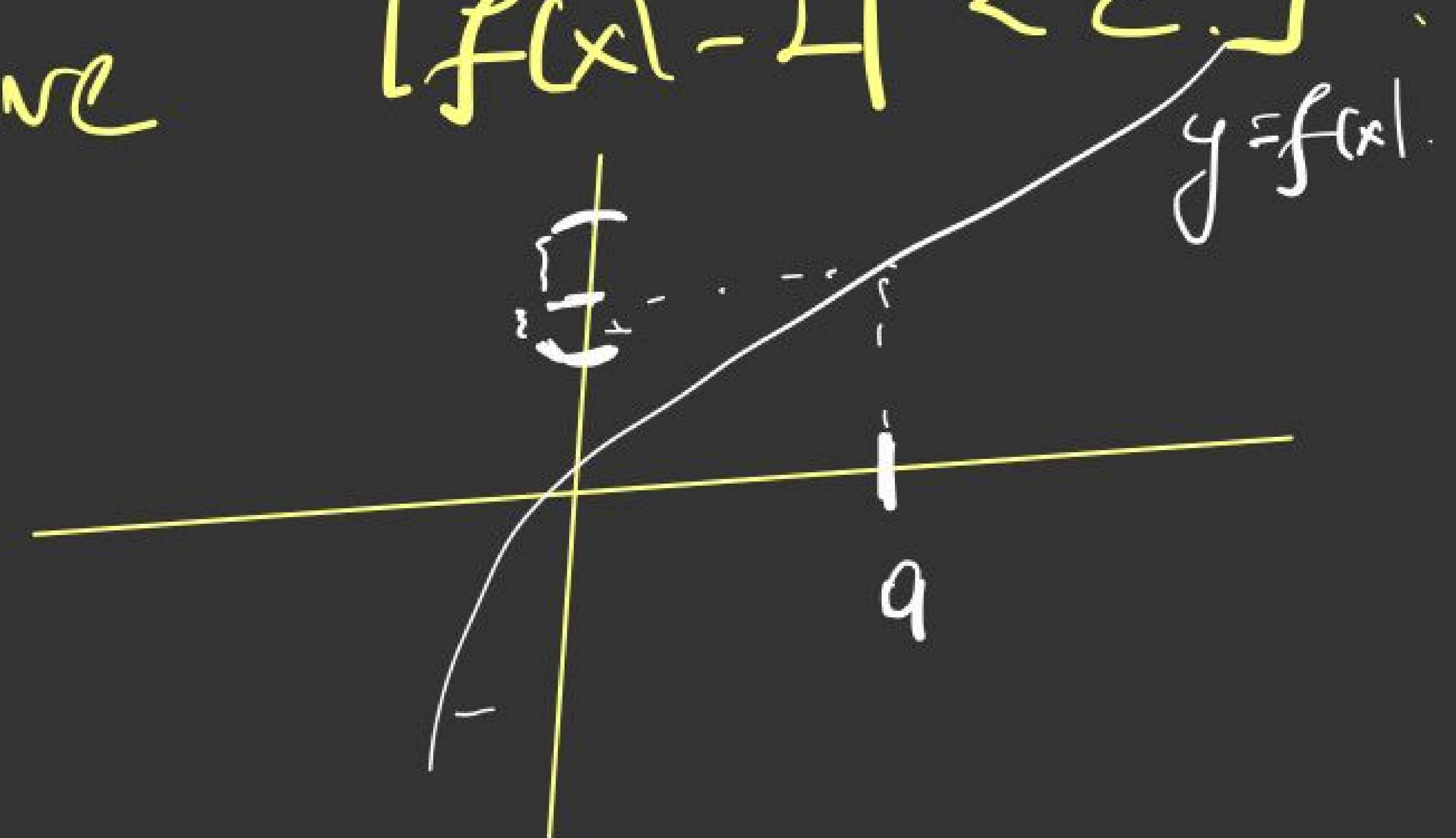
#5: Use the formal definition of the limit

to verify the following:

(a) $\lim\limits_{x \to c} \overbrace{(ax+b)}^{f(x)} = \overbrace{a \cdot c + b}^{L}$ for any $a, b, c \in \mathbb{R}$.

For all   there exists

$$\left[\ \lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \text{if } 0 < |x-a| < \delta, \text{ we have } |f(x) - L| < \varepsilon. \right]$$



Let $\varepsilon > 0$ be given. Then there exists

$\boxed{\delta = \frac{\varepsilon}{|a|}} > 0$ such that if

$0 < |x-c| < \delta$ we have

$|f(x) - L| = |ax+b-(a.c+b)| = |ax+\cancel{b}-ac-\cancel{b}|$

$= |ax - ac| = |a(x-c)| = |a| . \underbrace{|x-c|}_{<\delta}$

$< |a| . \delta \overset{\text{choose}}{=} \varepsilon$

$\delta = \frac{\varepsilon}{|a|}$ ∎.

(b) $\lim_{x \to 2} \overbrace{\frac{x-2}{1+x^2}}^{f(x)} = \overbrace{0}^{L}$.

Let $\varepsilon > 0$ be given. Then there exists some

$\boxed{\delta = \varepsilon} > 0$ such that if $0 < |x-2| < \delta$

then we have $|f(x) - L| = \left|\frac{x-2}{1+x^2} - 0\right| = \left|\frac{x-2}{1+x^2}\right|$

$= \frac{|x-2|}{1+x^2} = |x-2| . \underbrace{\frac{1}{1+x^2}}_{\leq 1} \leq |x-2| < \boxed{\delta = \varepsilon}$.

$\uparrow$ choose.

$1 + x^2 \geq 1$.

$\frac{1}{1+x^2} \leq 1$.

∎

(c) $\lim_{x \to 3} \sqrt{2x+3} = 3$. L

$f(x)$

Let $\varepsilon > 0$ be given. Then there exists

$\boxed{\delta = \frac{\varepsilon}{2}} > 0$ such that if

$0 < |x-3| < \delta$ we have

$$|f(x) - L| = \left|\sqrt{2x+3} - 3\right| = \frac{|(2x+3) - 9|}{|\sqrt{2x+3} + 3|}$$

$$= \frac{|2x-6|}{|\sqrt{2x+3}+3|} = \frac{2|x-3|^{<\delta}}{|\underbrace{\sqrt{2x+3}}_{\geq 0} + 3|} < 2 \cdot \underbrace{|x-3|}_{<\delta}$$

≥ 3

>1.

$< 2\delta = \varepsilon$

choose

$\boxed{\delta = \frac{\varepsilon}{2}}$