

STAT 271 / SUMMER 2007
MIDTERM EXAM I

1. Let X_n have the pdf

$$f_n(x) = \left[\frac{2n}{(3n+1)} \right], 0 \leq x \leq \frac{3n+1}{2n}, n = 1, 2, \dots$$

Find the limiting distribution of X_n , if any, by using the distribution function technique.
(25 pts.)

2. If $X_i \sim \text{Exp}(1), i = 1, 2, \dots, n$ all independent, then find the limiting distribution of $\sqrt{n}(\bar{X}_n - 1)$. (25 pts.)

3. If we have X_1, X_2, \dots, X_n i.i.d. random variables from the pdf

$$f(x; \theta) = \frac{4x^3}{\theta^2} \text{ where } 0 < x \leq \theta \text{ and } \theta > 0$$

- a) Find the method of moment estimator of θ . (5 pts.)
 - b) Find the maximum likelihood estimator of θ . (5 pts.)
 - c) Is MLE an unbiased estimator of θ ? Why or why not? (5 pts.)
 - d) Is MLE a consistent estimator of θ ? Why or why not? (5 pts.)
 - e) Find the unbiased estimator of θ^{2n+1} in terms of $\max(X_1, X_2, \dots, X_n)$. (5 pts.)
4. Let X_1, X_2, \dots, X_n be i.i.d. according to the following density

$$f(x; a) = \frac{a}{x^{a+1}}, 1 < x \text{ and } a > 0.$$

Find the MLE of $E(X)$. (25 pts.)

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1. (20 pts.) Let X_n have the p.d.f.

$$f_n(x; \theta) = n\theta e^{-n\theta x}, x > 0, \theta > 0, n = 1, 2, \dots$$

Find the limiting distribution of $Z_n = \frac{Y_1}{n^2}$ where Y_1 is the first order statistic.

$$F_n(x; \theta) = 1 - e^{-n\theta x}, x > 0, \theta > 0$$

$$G_n(y_1; \theta) = P(Y_1 \leq y_1) = 1 - [1 - F_n(y_1)]^n = 1 - e^{-n^2\theta y_1}, y_1 > 0, \theta > 0$$

$$H_n(z; \theta) = P(Z \leq z) = P\left(\frac{Y_1}{n^2} \leq z\right) = P(Y_1 \leq n^2 z) = 1 - e^{-n^4\theta z}, z > 0$$

$$\lim_{n \rightarrow \infty} H_n(z; \theta) = \lim_{n \rightarrow \infty} (1 - e^{-n^4\theta z}) = 1, z > 0 \Rightarrow \text{NOT A VALID DF}$$

Discontinuity point $z=0$.

$$H(z; \theta) = \begin{cases} 1, z \geq 0 \\ 0, z < 0 \end{cases} \Rightarrow \text{VALID DF}$$

Hence, the limiting distribution of $Z_n = \frac{Y_1}{n^2}$ is degenerate at $z=0$.

2. (20 pts.) If $X_i \sim \text{Exp}(1), i=1,2,\dots,n$ all independent, then find the limiting distribution of $\sqrt{n}(\bar{X}_n - 1)$ where \bar{X}_n is the sample mean.

$$X_i \sim \text{Exp}(1) \Rightarrow E(\bar{X}) = 1 \text{ and } V(\bar{X}) = 1/n$$

$$\text{The standardized sample mean: } \frac{\bar{X} - E(\bar{X})}{\sqrt{V(\bar{X})}} = \frac{\bar{X} - 1}{\sqrt{1/n}} = \sqrt{n}(\bar{X} - 1)$$

By CLT,

$$\frac{\bar{X} - E(\bar{X})}{\sqrt{V(\bar{X})}} = \sqrt{n}(\bar{X} - 1) \xrightarrow{d} N(0,1).$$

3. (20 pts.) Let (X_1, X_2, \dots, X_n) be a random sample from the following discrete distribution:

$$P(X_1 = 1) = \frac{2(1-\theta)}{2-\theta}, P(X_1 = 2) = \frac{\theta}{2-\theta}$$

where $\theta \in (0,1)$ is unknown. Obtain a moment estimator (MME) of θ .

$$\begin{aligned}\mu &= \sum_{x=1}^2 x.P(X=x) = 1.P(X=1) + 2.P(X=2) \\ &= \frac{2(1-\theta)}{2-\theta} + 2\frac{\theta}{2-\theta} = \frac{2-2\theta+2\theta}{2-\theta} = \frac{2}{2-\theta}\end{aligned}$$

MME of θ can be found by equating μ to \bar{X} .

$$\begin{aligned}\mu &= \bar{X} \\ \frac{2}{2-\theta} &= \bar{X} \\ 2 &= 2\bar{X} - \theta\bar{X} \\ \tilde{\theta} &= \frac{2\bar{X}-2}{\bar{X}} = 2 - \frac{2}{\bar{X}} \text{ is the MME of } \theta.\end{aligned}$$

4. Let (X_1, X_2, \dots, X_n) be a random sample from the uniform distribution on the interval

$(\theta, \theta + |\theta|)$. Find the MLE of θ when

(i) $\theta \in (0, \infty)$; (10 pts.)

$$\begin{aligned}f(x; \theta) &= \frac{1}{\theta + |\theta| - \theta} = \frac{1}{|\theta|} = \frac{1}{\theta}, \theta \leq x \leq 2\theta \text{ since } \theta \in (0, \infty). \\ L(\theta) &= \frac{1}{\theta^n}, \theta \leq x \leq 2\theta.\end{aligned}$$

The likelihood function is a decreasing function of θ . So, the minimum value of θ maximizes the likelihood function.

Let's find the range for θ .

$$\theta \leq y_1 < \dots < y_n \leq 2\theta \Rightarrow \begin{cases} \theta \leq y_1 \\ y_n \leq 2\theta \rightarrow \frac{y_n}{2} \leq \theta \Rightarrow \frac{y_n}{2} \leq \theta \leq y_1. \end{cases}$$

Since the minimum value of θ is $\frac{y_n}{2}$, the MLE of θ is $\hat{\theta} = \frac{Y_n}{2} = \frac{\max(X_1, \dots, X_n)}{2}$.

(ii) $\theta \in (-\infty, 0)$. (10 pts.)

$$f(x; \theta) = \frac{1}{\theta + |\theta| - \theta} = \frac{1}{|\theta|} = \frac{1}{-\theta}, \theta \leq x \leq 0 \text{ since } \theta \in (-\infty, 0).$$

$$L(\theta) = \frac{1}{\theta^n}, \theta \leq x \leq 0, -\infty < \theta < 0.$$

The likelihood function is a decreasing function of θ and θ is negative valued. So, the maximum value of θ maximizes the likelihood function.

The range of θ : $-\infty < \theta \leq y_1$.

Hence, MLE of θ is $\hat{\theta} = Y_1 = \min(X_1, \dots, X_n)$.

5. Let (X_1, X_2, \dots, X_n) be a random sample from a population with the pdf

$$f(x; \theta) = \frac{x^3}{6\theta^4} e^{-x/\theta}, x > 0, \theta > 0$$

where $\mu = E(X) = 4\theta$ and $\sigma^2 = \text{Var}(X) = 4\theta^2$.

f) Find the method of moment estimator (MME) of θ . (4 pts.)

$$\mu = \bar{X}$$

$$4\theta = \bar{X}$$

$$\tilde{\theta} = \frac{\bar{X}}{4} \text{ is an MME of } \theta.$$

g) Find the maximum likelihood estimator (MLE) of θ . (4 pts.)

$$L(\theta) = \frac{\prod_{i=1}^n x_i^3}{6^n \theta^{4n}} e^{-\sum_{i=1}^n x_i / \theta}, x_i > 0, \theta > 0$$

$$\ln L(\theta) = 3 \sum_{i=1}^n \ln x_i - n \ln 6 - 4n \ln \theta - \frac{\sum_{i=1}^n x_i}{\theta}$$

$$\frac{d \ln L(\theta)}{d\theta} = -\frac{4n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2} = 0 \rightarrow \hat{\theta} = \frac{\bar{X}}{4}$$

$$\left. \frac{d^2 \ln L(\theta)}{d\theta^2} \right|_{\theta=\hat{\theta}} < 0 \rightarrow \hat{\theta} = \frac{\bar{X}}{4} \text{ is maximizes the likelihood function.}$$

$$\Rightarrow \hat{\theta} = \frac{\bar{X}}{4} \text{ is the MLE of } \theta.$$

- h) Is MLE an unbiased estimator of θ ? Why or why not? If not, find an unbiased estimator of θ . (4 pts.)

$$E(\hat{\theta}) = E\left(\frac{\bar{X}}{4}\right) = \frac{E(\bar{X})}{4} = \frac{\mu}{4} = \frac{4\theta}{4} = \theta.$$

Thus, MLE of θ , $\hat{\theta} = \frac{\bar{X}}{4}$ an unbiased estimator of θ .

- i) Is MLE a consistent estimator of θ ? Why or why not? (4 pts.)

By Chebyshev's Inequality,

$$i) E(\hat{\theta}) = \theta$$

$$ii) V(\hat{\theta}) = V\left(\frac{\bar{X}}{4}\right) = \frac{V(\bar{X})}{16} = \frac{4\theta^2/n}{16} = \frac{\theta^2}{4n} < \infty \text{ for } \theta > 0.$$

$$iii) \lim_{n \rightarrow \infty} V(\hat{\theta}) = \lim_{n \rightarrow \infty} \frac{\theta^2}{4n} = 0.$$

All the conditions are satisfied. So, $\hat{\theta} \xrightarrow{p} \theta$. This means that $\hat{\theta}$ is a CE of θ .

- j) Find the MLE of μ and σ^2 . (4 pts.)

By the invariance property of MLE,

$$\hat{\mu} = 4\hat{\theta} = 4 \frac{\bar{X}}{4} = \bar{X} \text{ is an MLE of } \mu.$$

$$\hat{\sigma}^2 = 4\hat{\theta}^2 = 4 \left(\frac{\bar{X}}{4}\right)^2 = \frac{\bar{X}^2}{4} \text{ is an MLE of } \sigma^2.$$

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1. (20 pts.) Let $\{X_n\}$ be a sequence rvs with pmf

$$P(X_n = 0) = \frac{1}{n}, \quad P(X_n = 1) = \frac{1}{n+1}, \quad P(X_n = 2) = \frac{n-2}{n} \text{ and } P(X_n = 3) = \frac{1}{n(n+1)}$$

Find the limiting distribution of X_n , if exists.

$$F_n(x) = P(X_n \leq x) = \begin{cases} 0, & x < 0 \\ \frac{1}{n}, & 0 \leq x < 1 \\ \frac{2n+1}{n(n+1)}, & 1 \leq x < 2 \\ \frac{n^2+n-1}{n(n+1)}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

$$\lim_{n \rightarrow \infty} F_n(x) = \begin{cases} 0, & x < 2 \\ 1, & 2 \leq x \end{cases}$$

which is the cdf of the degenerate distribution at point 2. Hence the limiting distribution of X_n is the degenerate distribution at point 2.

2. (20 pts.) Let $X_n \sim \text{Exp}(\theta/n), \theta > 0, n=1,2,\dots$ with mgf

$$M_{X_n}(t) = \left(1 - \frac{\theta t}{n}\right)^{-1}, t < \frac{n}{\theta}$$

and $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ be the sample mean. Find the limiting distribution of \bar{X}_n by using the mgf technique. State the name of the limiting distribution.

$$M_{\bar{X}_n}(t) = \left(1 - \frac{\theta t}{n^2}\right)^{-n}, \frac{t}{n} < \frac{n}{\theta}$$

$$\lim_{n \rightarrow \infty} M_{\bar{X}_n}(t) = \lim_{n \rightarrow \infty} \left(1 - \frac{\theta t}{n^2}\right)^{-n} = e^{0t}, t < \infty \text{ where } \frac{\theta}{n^2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

which is the mgf of degenerate distribution at point 0. Hence, the limiting distribution of \bar{X}_n is degenerate at 0.

3. (20 pts.) Let X_1, X_2, \dots, X_n be i.i.d. r.v.s from $N(1,1)$ distribution. Find the limiting distribution of the r.v.

$$W_n = \sqrt{n} \frac{X_1 + X_2 + \dots + X_n - n}{(X_1 - 1)^2 + (X_2 - 1)^2 + \dots + (X_n - 1)^2}.$$

$$\text{i) } X \sim N(1,1) \Rightarrow \sum_{i=1}^n X_i \sim N(n,n) \Rightarrow \frac{\sum_{i=1}^n X_i - n}{\sqrt{n}} \sim N(0,1).$$

$$\text{ii) } X \sim N(1,1) \Rightarrow (X-1) \sim N(0,1) \Rightarrow (X-1)^2 \sim \chi_1^2 \Rightarrow Y = \sum_{i=1}^n (X_i-1)^2 \sim \chi_n^2$$

Then, by Chebyshev's Inequality,

$$\text{a) } E\left(\frac{Y}{n}\right) = 1$$

$$\text{b) } V\left(\frac{Y}{n}\right) = \frac{2}{n} < \infty \text{ for all } n.$$

$$\text{c) } \lim_{n \rightarrow \infty} V\left(\frac{Y}{n}\right) = 0$$

$$\frac{Y}{n} = \frac{\sum_{i=1}^n (X_i - 1)^2}{n} \xrightarrow{p} 1$$

By Slutsky's Theorem, $W \xrightarrow{d} N(0,1)$.

4. Let X_1, X_2, \dots, X_n be a random sample having pdf

$$f(x; \theta) = \frac{1}{2\theta+1}, \quad 0 \leq x \leq 2\theta+1, \quad \theta > 0 \Rightarrow X \sim \text{Uniform}(0, 2\theta+1)$$

a) (5 pts.) Find the method of moment estimator (MME) of θ .

$$E(X) = \frac{2\theta+1}{2} = \bar{X} \Rightarrow \tilde{\theta} = \frac{2\bar{X}-1}{2}$$

b) (5 pts.) Find the maximum likelihood estimator (MLE) of θ .

$$L(\theta) = \left(\frac{1}{2\theta+1} \right)^n, \quad 0 < x_i \leq \theta$$

$$\max_{\theta} L(\theta) \Rightarrow \min_{x_i \leq 2\theta+1} \theta \Rightarrow 0 < X_{(1)} \leq \dots \leq X_{(n)} \leq 2\theta+1$$

$$\Rightarrow \text{Minimum value of } \theta \text{ is } \frac{X_{(n)} - 1}{2} \Rightarrow \hat{\theta} = \frac{X_{(n)} - 1}{2}.$$

c) (5 pts.) Find the MLE of the median.

$$\int_0^{\text{median}} f(x; \theta) dx = 0.5 \Rightarrow \int_0^{\text{median}} \frac{1}{2\theta+1} dx = \frac{\text{median}}{2\theta+1} = 0.5 \Rightarrow \text{median} = \frac{2\theta+1}{2}.$$

$$\text{By the invariance property of MLE; } \hat{\text{median}} = \frac{2\hat{\theta}+1}{2} = \frac{2\left(\frac{X_{(n)}-1}{2}\right)+1}{2} = \frac{X_{(n)}}{2}.$$

d) (5 pts.) Find the mean squared errors (MSE) of the MLE and MME of θ and comment on which one is better estimator of θ .

To be able to find the MSE of the MLE, we need to find the pdf of $X_{(n)}$.

$$g_{Y_n}(y) = nF_X(y)^{n-1} f_X(y), -\infty < y < \infty$$

We also need the cdf of X.

$$F_X(x) = \int_0^x \frac{1}{2\theta+1} dx = \frac{x}{2\theta+1}, 0 < x \leq 2\theta+1$$

$$g_{X_{(1)}}(y) = n \frac{1}{2\theta+1} \left(\frac{y}{2\theta+1} \right)^{n-1} = \frac{ny^{n-1}}{(2\theta+1)^n}, 0 < y \leq 2\theta+1$$

$$E(X_{(1)}) = \int_0^{2\theta+1} y \frac{ny^{n-1}}{(2\theta+1)^n} dy = \frac{n}{n+1} (2\theta+1)$$

$$E(X_{(1)}^2) = \int_0^{2\theta+1} y^2 \frac{ny^{n-1}}{(2\theta+1)^n} dy = \frac{n}{n+2} (2\theta+1)^2$$

$$V(X_{(1)}) = \frac{n}{n+1} (2\theta+1)^2 - \left(\frac{n}{n+1} (2\theta+1) \right)^2 = \frac{n}{(n+2)(n+1)^2} (2\theta+1)^2$$

$$\begin{aligned}
MSE_{\theta}(\hat{\theta}) &= MSE_{\theta}\left(\frac{X(n)-1}{2}\right) = Var\left(\frac{X(n)-1}{2}\right) + \left(E\left(\frac{X(n)-1}{2}\right) - \theta\right)^2 \\
&= \frac{n}{4(n+2)(n+1)^2} (2\theta+1)^2 + \left(\frac{\left[\frac{n}{n+1}(2\theta+1)\right]-1}{2} - \theta\right)^2 \\
&= \frac{n(2\theta+1)^2}{4(n+2)(n+1)^2} + \left(\frac{-1-2\theta}{2(n+1)}\right)^2 \\
&= \frac{n(2\theta+1)^2}{4(n+2)(n+1)^2} + \frac{(2\theta+1)^2}{4(n+1)^2} \\
&= \frac{n(2\theta+1)^2 + (n+2)(2\theta+1)^2}{4(n+2)(n+1)^2} = \frac{(2\theta+1)^2}{2(n+2)(n+1)}
\end{aligned}$$

$$\begin{aligned}
MSE_{\theta}(\tilde{\theta}) &= MSE_{\theta}\left(\frac{2\bar{X}-1}{2}\right) = Var\left(\frac{2\bar{X}-1}{2}\right) + \left(E\left(\frac{2\bar{X}-1}{2}\right) - \theta\right)^2 \\
&= Var(\bar{X}) + \left(\frac{2\theta+1}{2} - \frac{1}{2} - \theta\right)^2 \\
&= \frac{(2\theta+1)^2}{n} + 0 = \frac{(2\theta+1)^2}{n}
\end{aligned}$$

$$\Rightarrow MSE_{\theta}(\hat{\theta}) < MSE_{\theta}(\tilde{\theta})$$

Hence, $\hat{\theta}$ is better estimator of θ .

5. Let X_1, X_2, \dots, X_n be a r.s. from the pdf

$$f(x; \alpha, \beta) = \frac{1}{\beta} e^{-\frac{(x-\alpha)}{\beta}}, \alpha \leq x < \infty, -\infty < \alpha < \infty, \beta > 0.$$

- (5 pts.) Find the MLE of α and β .

$$L(\alpha, \beta) = \frac{1}{\beta^n} e^{-\frac{\sum_{i=1}^n (x_i - \alpha)}{\beta}}, \alpha \leq x_i < \infty$$

$$\ln L(\alpha, \beta) = -n \ln \beta - \frac{\sum_{i=1}^n (x_i - \alpha)}{\beta}, \alpha \leq x_i < \infty$$

$$\frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = 0 \Rightarrow \text{No solution for } \alpha.$$

$$\max_{\alpha} L(\alpha, \beta) \Rightarrow \max_{\alpha \leq x_i} \alpha \Rightarrow \alpha \leq X_{(1)} \leq \dots \leq X_{(n)} < \infty \Rightarrow \alpha \leq X_{(1)}$$

The maximum value of α is $X_{(1)}$. Hence, $\hat{\alpha} = X_{(1)}$.

$$\frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = -\frac{n}{\beta} + \frac{\sum_{i=1}^n (X_i - \alpha)}{\beta^2} = 0 \Rightarrow \hat{\beta} = \frac{\sum_{i=1}^n (X_i - \hat{\alpha})}{n} = \frac{\sum_{i=1}^n (X_i - X_{(1)})}{n}$$

$$\left. \frac{\partial^2 \ln L(\alpha, \beta)}{\partial \beta^2} \right|_{\beta=\hat{\beta}} < 0 \Rightarrow \hat{\beta} \text{ maximizes the likelihood function.}$$

- **(5 pts.)** Find the MLE of $P(X_1 \geq 1)$ where X_1 is one observation not an order statistic.

$$P(X_1 \geq 1) = \int_1^{\infty} \frac{1}{\beta} e^{-\frac{(x_1 - \alpha)}{\beta}} dx_1 = e^{-\frac{(1 - \alpha)}{\beta}}.$$

$$\text{By the invariance property of MLE, } P(X_1 \geq 1) = e^{-\frac{(1 - \hat{\alpha})}{\hat{\beta}}} = e^{-\frac{n(1 - X_{(1)})}{\sum_{i=1}^n (X_i - X_{(1)})}}.$$

- **(5 pts.)** Is the MLE of α an unbiased estimator of α ?

$$E(X_{(1)}) \stackrel{?}{=} \theta$$

$$F_X(x) = 1 - e^{-\frac{(x-\alpha)}{\beta}}, \alpha \leq x < \infty$$

$$g_{X(1)}(y) = n f_X(y) [1 - F_X(y)]^{n-1} = \frac{n}{\beta} e^{-\frac{n(y-\alpha)}{\beta}}, \alpha \leq y < \infty$$

$$E(X_{(1)}) = \int_{\alpha}^{\infty} y \frac{n}{\beta} e^{-\frac{n(y-\alpha)}{\beta}} dy = \alpha - \frac{\beta}{n} \neq \alpha$$

$\hat{\alpha} = X_{(1)}$ is not an unbiased estimator of α .

- **(5 pts.)** Is the MLE of α a consistent estimator of α ?

By the Chebyshev's Inequality,

$$\text{i) } E(X_{(1)}) = \alpha - \frac{\beta}{n} \Rightarrow \lim_{n \rightarrow \infty} E(X_{(1)}) = \lim_{n \rightarrow \infty} \left(\alpha - \frac{\beta}{n} \right) = \alpha$$

$$\text{ii) } E(X_{(1)}^2) = \int_{\alpha}^{\infty} y^2 \frac{n}{\beta} e^{-\frac{n(y-\alpha)}{\beta}} dy = \alpha^2 - \frac{2\beta}{n} \left(\alpha - \frac{\beta}{n} \right)$$

$$\text{Var}(X_{(1)}) = \alpha^2 - \frac{2\beta}{n} \left(\alpha - \frac{\beta}{n} \right) - \left(\alpha - \frac{\beta}{n} \right)^2 = \frac{\beta^2}{n} < \infty$$

$$\text{iii) } \lim_{n \rightarrow \infty} \text{Var}(X_{(1)}) = 0$$

This means that $X_{(1)} \xrightarrow{P} \alpha$ and $X_{(1)}$ is a CE of α .

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MIDTERM EXAM II

1. Consider a random sample of size n from a distribution with p.d.f.

$$f(x, \theta) = \frac{1}{\theta}, 0 < x \leq \theta, \theta > 0$$

where $\mu = \frac{\theta}{2}$ and $\sigma^2 = \frac{\theta^2}{12}$.

The MLE, $\hat{\theta} = \max(X_1, \dots, X_n) = Y_n$ and the MME, $\tilde{\theta} = 2\bar{X}$.

- a) Is MLE an unbiased estimator of θ ? (10 pts.)
- b) Is MME an unbiased estimator of θ ? (10 pts.)
- c) Compare MSEs of MLE and MME of θ ? (15 pts.)

p.d.f. of Y_n : $g(y_n, \theta) = n [F_X(y_n)]^{n-1} f_X(y_n), a < y_n < b$

2. Consider the p.d.f.

$$f(x, \theta) = \theta x^{\theta-1} e^{-x}, \quad 0 < x, \theta > 0$$

Find a sufficient statistics for θ . (30 pts.)

3. Let X have the p.d.f.

$$f(x, \theta) = \frac{2x}{\theta^2}, \quad 0 < x < \theta, \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

- a) Find a sufficient statistic Y for θ by using the factorization theorem. (10 pts.)
- b) Show that Y is a complete sufficient statistic for θ . (15 pts.)
- c) Find the unique MVUE of θ . (10 pts.)

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1. (20 pts.) Let X have the p.d.f.

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \theta > 0$$

with $\mu = \theta$ and $\sigma^2 = \theta^2$. Consider a r.s. of size 5; X_1, X_2, \dots, X_5 . Let

$$Y_1 = \bar{X} = \frac{\sum_{i=1}^5 X_i}{5} \text{ and } Y_2 = \frac{X_1 + X_2}{5}.$$

Find the mean squared errors of Y_1 and Y_2 . Which one is better estimator? Why?

$$E(Y_1) = \mu = \theta \text{ and } V(Y_1) = \frac{\sigma^2}{n} = \frac{\theta^2}{5} = 0.2\theta^2 = MSE_{\theta}(Y_1)$$

$$E(Y_2) = \frac{2}{5}\theta \text{ and } V(Y_2) = \frac{2}{25}\theta^2$$

$$\begin{aligned} MSE_{\theta}(Y_2) &= V(Y_2) + [E(Y_2) - \theta]^2 = \frac{2}{25}\theta^2 - \left[\frac{2}{5}\theta - \theta\right]^2 \\ &= \frac{2}{25}\theta^2 + \frac{9}{25}\theta^2 = \frac{11}{25}\theta^2 = 0.44\theta^2 \end{aligned}$$

Since $MSE_{\theta}(Y_1) = 0.2\theta^2 < MSE_{\theta}(Y_2) = 0.44\theta^2$, Y_1 is a better estimator of θ .

2. Let (X_1, \dots, X_n) be a random sample from the distribution on R with pdf

$$f(x; a) = \theta a^{\theta} x^{-(\theta+1)}, \quad a \leq x,$$

where $a > 0$ and θ is known and different than 0.

a) (15 pts.) Find a complete sufficient statistic for a .

$$L(a) = \theta^n a^{n\theta} \prod_{i=1}^n x_i^{-(\theta+1)}, \quad a \leq x_i$$

Given $Y_1 = \min(X_1, \dots, X_n)$, the conditional range of x_i is $y_1 \leq x_i$ and

$$L(a) = \underbrace{\theta^n a^{n\theta} \prod_{i=1}^n x_i^{-(\theta+1)} \frac{y_1}{y_1}}_{k_1(y_1; a)} \cdot \frac{1}{k_2(x_1, \dots, x_n)}, \quad y_1 \leq x_i$$

By Neyman's Factorization theorem, $Y_1 = \min(X_1, \dots, X_n)$ is a s.s. for a .

$$F(x; a) = \int_a^{\infty} \theta a^{\theta} x^{-(\theta+1)} dx = 1 - a^{\theta} x^{-\theta}, x \geq a$$

$$g_{Y_1}(y) = n\theta a^{n\theta} y_1^{-n\theta-1}, y_1 \geq a$$

$$\begin{aligned} E(u(Y_1)) &= \int_a^{\infty} u(y_1) n\theta a^{n\theta} y_1^{-n\theta-1} dy_1 = 0 \Rightarrow \int_a^{\infty} u(y_1) y_1^{-n\theta-1} dy_1 = 0 \\ &\Rightarrow \frac{d}{da} \int_a^{\infty} u(y_1) y_1^{-n\theta-1} dy_1 = 0 \end{aligned}$$

Now, apply Leibnitz's Rule:

$$\frac{d}{da} \int_a^{\infty} u(y_1) y_1^{-n\theta-1} dy_1 = 0 - u(a) a^{-n\theta-1} + \underbrace{\int_a^{\infty} \frac{\partial}{\partial a} u(y_1) y_1^{-n\theta-1} dy_1}_{0 \text{ since the function does not depend on } a} = 0$$

$$u(a) \underbrace{a^{-n\theta-1}}_{\neq 0} = 0 \Rightarrow u(a) = 0 \text{ for all } a > 0 \Rightarrow u(y_1) = 0 \text{ for all } y_1 > 0 \Rightarrow u(y_1) = 0 \text{ for all } y_1 \geq a$$

Hence Y_1 is a c.s.s. for θ .

b) (10 pts.) Obtain the unique minimum variance unbiased estimator of a , if exists.

$$E(Y_1) = \int_a^{\infty} y_1 n\theta a^{n\theta} y_1^{-n\theta-1} dy_1 = n\theta a^{n\theta} \left. \frac{y_1^{-n\theta+1}}{-n\theta+1} \right|_a^{\infty} = \frac{n\theta}{n\theta-1} a$$

$$E\left(\frac{n\theta-1}{n\theta} Y_1\right) = a$$

Hence, $\frac{n\theta-1}{n\theta} Y$ is the unique UMVUE of a .

3. (20 pts.) Let Y be complete sufficient statistics for the unknown parameter θ and a function of sample random variables; $u(X_1, X_2, \dots, X_n)$ is an unbiased estimator of $m(\theta)$; a function of θ . How can we find the unique minimum variance unbiased estimator of $m(\theta)$ by using $u(X_1, X_2, \dots, X_n)$ and Y ? Explain. Which theorems did you use to answer the question?

$$E[u(X_1, X_2, \dots, X_n)] = m(\theta).$$

Let $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ be a function of Y only.

By **Rao-Blackwell Theorem**;

$$E = \{\varphi(Y)\} = E\left\{E[u(X_1, X_2, \dots, X_n) | Y]\right\} = m(\theta).$$

Hence, $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ is an UE of $m(\theta)$.

Since Y is complete sufficient statistics for θ and $u(X_1, X_2, \dots, X_n)$ is an UE of $m(\theta)$, $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ is an $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ and a function of Y. By **Lehmann-Scheffe Theorem**, $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ is the unique MVUE of $m(\theta)$.

4. Let X have the p.d.f.

$$f(x; \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}, -\infty < x < \infty, -\infty < \mu < \infty$$

where $\sigma^2 = 1$. Assume that we have a random sample of size n.

a) (10 pts.) Find a complete sufficient statistics for μ .

$$f(x; \mu) = e^{-\frac{1}{2} \ln(2\pi) - \frac{x^2 - 2x\mu + \mu^2}{2}}, \underbrace{-\infty < x < \infty}_{\text{continuous r.v.}}, \underbrace{-\infty < \mu < \infty}_{\text{an interval}}$$

$$P(\mu) = \mu, K(x) = x, S(x) = -\frac{x^2}{2}, q(\mu) = -\frac{1}{2} \ln(2\pi) - \frac{\mu^2}{2}.$$

This pdf belongs to exponential class of pdfs of continuous type.

Regularity conditions:

- a) Range of x does not depend on μ .
- b) $P(\mu) = \mu$ is a nontrivial, continuous function of μ for $-\infty < \mu < \infty$.
- c) $K'(x) = 1$ is a nontrivial, continuous function of x for $-\infty < x < \infty$.
- d) $S(x) = -\frac{x^2}{2}$ is a continuous function of x for $-\infty < x < \infty$.

For a r.s., $Y = \sum_{i=1}^n K(X_i) = \sum_{i=1}^n X_i$ is a c.s.s. for μ .

b) (5 pts.) Find the unique minimum variance unbiased estimator of μ .

$$E(Y) = E\left(\sum_{i=1}^n X_i\right) = n\mu \Rightarrow E(\bar{X}) = \mu.$$

By Lehmann-Scheffe Theorem, \bar{X} is the unique minimum variance unbiased estimator of μ .

c) (15 pts.) Find the unique minimum variance unbiased estimator of $e^{\mu t}$ where $-\infty < t < \infty$.

$$E(e^{\bar{X}t}) = e^{\mu t + \frac{t^2}{2n}} \Rightarrow \text{mgf of } \bar{X} \sim N(\mu, 1/n) \Rightarrow E\left(e^{\bar{X}t - \frac{t^2}{2n}}\right) = e^{\mu t}$$

By Lehmann-Scheffe Theorem, $e^{\bar{X}t - \frac{t^2}{2n}}$ is the unique minimum variance unbiased estimator of $e^{\mu t}$.

d) (5 pts.) Find $E\left[\frac{X_1}{n} \mid \bar{X}\right]$.

$$E\left\{E\left[\frac{X_1}{n} \mid \bar{X}\right]\right\} = E\left(\frac{X_1}{n}\right) = \frac{\mu}{n}.$$

Since \bar{X} is a c.s.s. for μ , there is only one function of \bar{X} whose expectation is $\frac{\mu}{n}$.

$$\text{Hence, } E\left[\frac{X_1}{n} \mid \bar{X}\right] = \frac{\bar{X}}{n}.$$

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MIDTERM EXAM 2

4. (15 pts.) Consider the p.d.f.

$$f(x; \theta) = \theta x^{\theta-1} e^{-x}, x > 0, \theta > 0$$

Assuming that we have a random sample of size n , find a sufficient statistics for θ using the Neyman's factorization theorem.

$$L(\theta) = \underbrace{\theta^n \prod_{i=1}^n x_i^{\theta-1}}_{k_1\left(y = \prod_{i=1}^n x_i; \theta\right)} \underbrace{e^{-\sum_{i=1}^n x_i}}_{k_2(x_1, x_2, \dots, x_n)}, \quad \underbrace{x_i > 0}_{\text{not depend on } \theta}$$

$Y = \prod_{i=1}^n X_i$ is a sufficient statistics for θ .

5. Let X have the p.d.f.

$$f(x; \theta) = 2\theta^2 x^{-3}, \theta \leq x, \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

d) (8 pts.) Find a sufficient statistic, Y for θ by using the Neyman's factorization theorem.

e) (8 pts.) Show that Y is a complete sufficient statistic for θ .

f) (9 pts.) Find the unique MVUE of θ . (10 pts.)

- PDF of $X_{(n)}$: $g_{X_{(n)}} = n f_X(x_{(n)}) [F_X(x_{(n)})]^{n-1}, -\infty < x_{(n)} < \infty$
- PDF of $X_{(1)}$: $g_{X_{(1)}} = n f_X(x_{(1)}) [1 - F_X(x_{(1)})]^{n-1}, -\infty < x_{(1)} < \infty$
- Leibnitz's Rule:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x; t) dx = f(b(t); t) \left(\frac{d}{dt} [b(t)] \right) - f(a(t); t) \left(\frac{d}{dt} [a(t)] \right) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x; t) dx$$

$$a) \quad L(\theta) = 2^n \theta^{2n} \frac{x_{(1)}}{\underbrace{\prod_{i=1}^n x_i^{-3}}_{\substack{\text{conditional range of } x_i \text{ given } x_{(1)} \\ x_{(1)} \leq x_i}}}, \quad \theta \leq x_i, \quad \theta > 0$$

$Y = X_{(1)}$ is a ss for θ .

$$b) \quad F(x; \theta) = \int_{\theta}^x 2\theta^2 x^{-3} dx = 1 - \frac{\theta^2}{x^2}, \quad \theta \leq x$$

$$g_{X_{(1)}} = n f_X(x_{(1)}) [1 - F_X(x_{(1)})]^{n-1}, \quad -\infty < x_{(1)} < \infty$$

$$= n 2\theta^2 x_{(1)}^{-3} \left[1 - \left(1 - \frac{\theta^2}{x^2} \right) \right]^{n-1}$$

$$= 2n\theta^{2n} x_{(1)}^{-2n-1}, \quad \theta \leq x_{(1)}$$

$X_{(1)}$ is complete iff $E[U(X_{(1)})] = 0$ for $\theta > 0$ implies $U(X_{(1)}) = 0$ for all $\theta \leq x$.

$$E[U(X_{(1)})] = \int_{\theta}^{\infty} u(x_{(1)}) 2n\theta^{2n} x_{(1)}^{-2n-1} dx = 0$$

$$\Rightarrow \int_{\theta}^{\infty} u(x_{(1)}) x_{(1)}^{-2n-1} dx = 0$$

$$\Rightarrow \frac{d}{d\theta} \int_{\theta}^{\infty} u(x_{(1)}) x_{(1)}^{-2n-1} dx = 0$$

Now apply Leibnitz rule,

$$\frac{d}{d\theta} \int_{\theta}^{\infty} u(x_{(1)}) x_{(1)}^{-2n-1} dx = 0 - u(\theta) \theta^{-2n-1} + \underbrace{\int_{\theta}^{\infty} \frac{\partial u(x_{(1)}) x_{(1)}^{-2n-1}}{\partial \theta} dx}_{0} = 0$$

$$\Rightarrow \underbrace{-u(\theta) \theta^{-2n-1}}_{\neq 0} = 0$$

$$\Rightarrow u(\theta) = 0 \text{ for } \theta > 0$$

$$\Rightarrow u(x_{(1)}) = 0 \text{ for } x > 0$$

$$\Rightarrow u(x_{(1)}) = 0 \text{ for } x \geq \theta$$

Hence, $Y = X_{(1)}$ is complete and combining a) and b) we can say that $Y = X_{(1)}$ is css for θ .

c)

$$E(X_{(1)}) = \int_{\theta}^{\infty} x_{(1)} 2n \theta^{2n} x_{(1)}^{-2n-1} dx_{(1)} = 2n \theta^{2n} \frac{x_{(1)}^{-2n+1}}{-2n+1} \Big|_{\theta}^{\infty} = \frac{2n}{2n-1} \theta$$

$$\Rightarrow E\left(\frac{2n-1}{2n} X_{(1)}\right) = \theta$$

Hence, $\frac{2n-1}{2n} X_{(1)}$ is the MVUE of θ .

3. Let X_1, X_2, \dots, X_n be a r.s. from $\text{Ber}(p)$ distribution with pdf

$$f(x; p) = p^x (1-p)^{1-x}, x = 0, 1; 0 < p < 1$$

- a) (10 pts.) Find the MVUE of $\text{Var}(X) = p(1-p)$
- b) (10 pts.) Find the MVUE of p^2 .

$$f(x; p) = p^x (1-p)^{1-x}, x = 0, 1; 0 < p < 1$$

$$= \left(\frac{p}{1-p}\right)^x (1-p) = e^{x \ln \frac{p}{1-p} + \ln(1-p)}$$

$$P(p) = \ln \frac{p}{1-p}, K(x) = x, q(p) = \ln(1-p), S(x) = 0$$

The pdf is a member of exponential class of pdfs of discrete type.

Conditions on regular case:

- i) Range of x does not depend on p .
- ii) $P(p)$ is a non-trivial continuous function of p , $0 < p < 1$.
- iii) $K(x) = x$ is a nontrivial function of x .

All the conditions are satisfied, so we have a regular case exponential class of pdfs of discrete type.

For a r.s, $Y = \sum_{i=1}^n K(X_i) = \sum_{i=1}^n X_i$ is a css for p .

a)

$$E(Y^2) = V(Y) + E^2(Y) = np(1-p) + n^2 p^2$$

$$\Rightarrow E\left(\frac{Y^2}{n}\right) = p(1-p) + np^2$$

$$E(Y) = np$$

$$E\left(Y - \frac{Y^2}{n}\right) = np - p(1-p) - np^2 = np(1-p) - p(1-p) = (n-1)p(1-p)$$

$$\Rightarrow E\left(\frac{1}{n-1}\left(Y - \frac{Y^2}{n}\right)\right) = p(1-p)$$

Hence, $\frac{1}{n-1}\left(Y - \frac{Y^2}{n}\right)$ is the MVUE of $p(1-p)$.

$$\text{b) } E(Y^2) = V(Y) + E^2(Y) = np(1-p) + n^2 p^2 = np - np^2 + n^2 p^2$$

$$E(Y) = np$$

$$E(Y^2 - Y) = np - np^2 + n^2 p^2 - np = (n^2 - n)p^2$$

$$E\left(\frac{Y^2 - Y}{n^2 - n}\right) = p^2$$

$\frac{Y^2 - Y}{n^2 - n}$ is the MVUE of p .

4. Let X_1, X_2, \dots, X_n be a r.s. from the distribution with the following pdf

$$f(x; \theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta}, 0 < x < \infty, \theta > 0$$

a) (6 pts.) Write the pdf in the exponential form.

b) (6 pts.) If X_1, X_2, \dots, X_n is a r.s from this pdf, find a complete sufficient statistic for θ

c) (8 pts.) Find the MVUE of θ .

[Hint: $X \sim \text{Gamma}(4, \theta)$. So, $E(X) = 4\theta$ and $\text{Var}(X) = 4\theta^2$.]

$$\text{a) } f(x; \theta) = e^{-\ln 6 - 4 \ln \theta + 3 \ln x - x/\theta}, 0 < x < \infty, \theta > 0$$

$$P(\theta) = -\frac{1}{\theta}, K(x) = x, q(\theta) = 4 \ln(\theta), S(x) = -\ln 6 + 3 \ln x$$

The pdf is a member of exponential class of pdfs of continuous type.

b) Conditions on regular case:

- i) Range of x does not depend on θ .
- ii) $P(\theta)$ is a non-trivial continuous function of θ , $0 < \theta$.
- iii) $K'(x) = 1$ not identical to 0 and $K(x)$ is continuous function of x , $x > 0$.
- iv) $S(x)$ is a continuous function of x , $x > 0$

All the conditions are satisfied, so we have a regular case exponential class of pdfs of discrete type.

For a r.s, $Y = \sum_{i=1}^n K(X_i) = \sum_{i=1}^n X_i$ is a css for θ .

$$\text{iii) } E(Y) = E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n 4\theta = 4n\theta$$
$$\text{So, } E\left(\frac{Y}{4n}\right) = \theta.$$

Therefore, $Y/(4n)$ is the MVUE of θ .

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FINAL EXAM

2. Let \bar{X}_n denote the mean of a r.s. of size n from $\text{Gamma}(\alpha, 1)$. Show that the limiting distribution of $\sqrt{n}(\bar{X}_n - \alpha) / \sqrt{\bar{X}_n}$ is $N(0, 1)$. (25 pts.)

3. Suppose that X is a r.v. with p.d.f.

$$f(x, \theta) = \frac{1}{\theta} e^{-\frac{(x-\alpha)}{\theta}}, \alpha \leq x < \infty, \alpha > 0, \theta > 0$$

- a) Find a joint sufficient statistics for (α, θ) by factorization theorem. (5 pts.)
- b) Are the j.s.s. for (α, θ) found in a) also minimal joint sufficient statistics? Why? (5 pts.)
- c) It is known that $Y_1 = \min\{X_1, \dots, X_n\}$ and $\sum_{i=1}^n X_i$ are joint complete sufficient statistics for (α, θ) . Find the unique minimum variance unbiased estimator of θ . (15 pts.)

4. Let X have the p.d.f.

$$f(x, \beta) = \frac{1}{\beta^2} x e^{-x/\beta}, x > 0, \beta > 0$$

where $\mu = 2\beta$ and $\sigma^2 = 2\beta^2$.

a) Find the unique minimum variance unbiased estimator of β^2 . (15 pts.)

b) Find $E[X_1 | \bar{X}]$. (10 pts.)

5. Let $X \sim \text{Geo}(p)$ where $f(x, p) = p(1-p)^{x-1}; x=1, 2, \dots; 0 < p < 1$ where $\mu = 1/p$.

a) Find the MLE of p . (5 pts.)

b) Find the Fisher Information in a random variable. (5 pts.)

c) Find the Rao-Cramer Lower Bound for $1/p$. (5 pts.)

d) Find the efficient estimator of p , if it exists. (5 pts.)

e) Find the efficient estimator of μ , if it exists. If it does not exist, find the asymptotic efficient estimator of μ , and specify its asymptotic distribution. (5 pts.)

STAT 271 / FALL 2007

FINAL EXAM

1. (25 pts.) Let X_i be the weight of the i -th airline passenger's luggage. Assume that weights are independent and each with pdf

$$f(x; \theta) = \theta 2^{-\theta} x^{\theta-1}, 0 < x < 2$$

and zero otherwise. Find the limiting distribution of $\frac{n^{1/\theta} Y_1}{2}$ where Y_1 is the first order statistic.

$$F_X(x; \theta) = \int_0^x \theta 2^{-\theta} w^{\theta-1} dw = \left(\frac{x}{2}\right)^\theta, 0 < x < 2$$

$$G_{Y_1}(y; \theta) = P(Y_1 \leq y) = 1 - [1 - F_X(y)]^n, 0 < y < 2$$

$$= 1 - \left[2 - \left(\frac{y}{2}\right)^\theta\right]^n, 0 < y < 2$$

$$\text{Let } Z_n = \frac{n^{1/\theta} Y_1}{2}.$$

$$\begin{aligned}
H_{Z_n}(z; \theta) &= P(Z_n \leq z) = P\left(\frac{n^{1/\theta} Y_1}{2} \leq z\right) = P\left(Y_1 \leq \frac{2z}{n^{1/\theta}}\right) \\
&= 1 - \left[1 - \left(\frac{\frac{2z}{n^{1/\theta}}}{2}\right)^\theta\right]^n = 1 - \left[1 - \frac{z^\theta}{n}\right]^n, 0 < z < n^{1/\theta} \\
H_Z(z; \theta) &= \lim_{n \rightarrow \infty} H_{Z_n}(z; \theta) = \lim_{n \rightarrow \infty} \left(1 - \left[1 - \frac{z^\theta}{n}\right]^n\right) = 1 - e^{-z^\theta}, z \geq 0
\end{aligned}$$

2. Suppose that X is a r.v. with p.d.f.

$$f(x; \alpha, \beta) = \frac{1}{2\alpha}, \beta - \alpha \leq x \leq \beta + \alpha, -\infty < \beta < \infty, \alpha > 0 \Rightarrow \text{Uniform}(\beta - \alpha, \beta + \alpha)$$

Assuming that X_1, \dots, X_n is a random sample, each X_i is distributed as the above pdf. It is known that $Y_1 = \min\{X_1, \dots, X_n\}$ and $Y_n = \max\{X_1, \dots, X_n\}$ are joint complete sufficient statistics for (α, β) .

- (10 pts.) Find the unique minimum variance unbiased estimators of α .
- (10 pts.) Find the unique minimum variance unbiased estimators of β .
- (15 pts.) Find the unique minimum variance unbiased estimators of β/α .

$$F_X(x; \alpha, \beta) = \frac{x - \beta + \alpha}{2\alpha}, \beta - \alpha \leq x \leq \beta + \alpha$$

$$\begin{aligned}
E(Y_1) &= \int_{\beta-\alpha}^{\beta+\alpha} y_1 n \left(1 - \frac{y_1 - \beta + \alpha}{2\alpha}\right)^{n-1} \frac{1}{2\alpha} dy_1 = \frac{n}{2^n \alpha^n} \int_{\beta-\alpha}^{\beta+\alpha} y_1 (\beta + \alpha - y_1)^{n-1} dy_1 \\
&= \frac{n}{2^n \alpha^n} \int_0^{2\alpha} (\beta + \alpha - u) u^{n-1} du \\
&= \frac{n}{2^n \alpha^n} \left[(\alpha + \beta) \frac{2^n \alpha^n}{n} - \frac{2^{n-1} \alpha^{n-1}}{n-1} \right] \\
&= (\alpha + \beta) - \frac{2n\alpha}{n+1}
\end{aligned}$$

$$\begin{aligned}
E(Y_n) &= \int_{\beta-\alpha}^{\beta+\alpha} y_n n \left(\frac{y_n - \beta + \alpha}{2\alpha} \right)^{n-1} \frac{1}{2\alpha} dy_1 = \frac{n}{2^n \alpha^n} \int_{\beta-\alpha}^{\beta+\alpha} y_n (y_n - \beta + \alpha)^{n-1} dy_n \\
&= \frac{n}{2^n \alpha^n} \int_0^{2\alpha} (u - \beta + \alpha) u^{n-1} du \\
&= \frac{n}{2^n \alpha^n} \left[(\beta - \alpha) \frac{2^n \alpha^n}{n} - \frac{2^{n-1} \alpha^{n-1}}{n-1} \right] \\
&= (\beta - \alpha) + \frac{2n\alpha}{n+1}
\end{aligned}$$

$$\mathbf{a)} \ E(Y_n - Y_1) = \frac{2(n-1)}{n+1} \alpha \Rightarrow E\left(\frac{n+1}{2(n-1)}(Y_1 + Y_n)\right) = \alpha.$$

Hence, $\frac{n+1}{2(n-1)}(Y_1 + Y_n)$ is the unique minimum variance unbiased estimators of α .

$$\mathbf{b)} \ E(Y_1 + Y_n) = 2\beta \Rightarrow E\left(\frac{Y_1 + Y_n}{2}\right) = \beta.$$

Hence, $\frac{Y_1 + Y_n}{2}$ is the unique minimum variance unbiased estimators of β .

$$\mathbf{c)} \text{ Consider } \frac{Y_n + Y_1}{Y_n - Y_1}.$$

$$\begin{aligned}
E\left(\frac{Y_n + Y_1}{Y_n - Y_1}\right) &= \int_{\beta-\alpha}^{\beta+\alpha} \int_{y_1}^{\beta+\alpha} \frac{y_n + y_1}{y_n - y_1} \underbrace{\frac{n(n-1)}{2^n \alpha^n} [y_n - y_1]^{n-2}}_{\text{joint pdf of } Y_1 \text{ and } Y_n} dy_n dy_1 \\
&= \frac{n(n-1)}{2^n \alpha^n} \int_{\beta-\alpha}^{\beta+\alpha} \int_{y_1}^{\beta+\alpha} (y_n + y_1) [y_n - y_1]^{n-3} dy_n dy_1 \\
&\stackrel{u=y_n-y_1}{=} \frac{n(n-1)}{2^n \alpha^n} \int_{\beta-\alpha}^{\beta+\alpha} \int_0^{\beta+\alpha-y_1} (2y_1 + u) [u]^{n-3} du dy_1 \\
&= \frac{n(n-1)}{2^n \alpha^n} \int_{\beta-\alpha}^{\beta+\alpha} \left\{ \frac{2}{n-2} y_1 (\alpha + \beta - y_1)^{n-2} + \frac{1}{n-1} (\alpha + \beta - y_1)^{n-1} \right\} dy_1 \\
&\stackrel{u=\alpha+\beta-y_1}{=} \frac{n(n-1)}{2^n \alpha^n} \left[\frac{2}{n-2} \int_0^{2\alpha} (\alpha + \beta - u) u^{n-2} du + \frac{1}{n-1} \frac{2^n \alpha^n}{n} \right] \\
&= \frac{n(n-1)}{2^n \alpha^n} \left[\frac{2}{n-2} (\alpha + \beta) \frac{2^{n-1} \alpha^{n-1}}{n-1} + - \frac{2}{n-2} \frac{2^n \alpha^n}{n} \frac{1}{n-1} \frac{2^n \alpha^n}{n} \right] \\
&= \frac{n}{n-2} \frac{\beta}{\alpha}
\end{aligned}$$

Hence, $\frac{n-2}{n} \frac{Y_n + Y_1}{Y_n - Y_1}$ unique minimum variance unbiased estimators of β/α .

3. Let $X \sim Poi(\theta)$ where $f(x; \theta) = \frac{e^{-\theta} \theta^x}{x!}; x = 0, 1, \dots; 0 < \theta$ where $\mu = \theta$.

f) (5 pts.) Find the MLE of θ .

$$L(\theta) = \frac{e^{-n\theta} \theta^{n\bar{x}}}{\prod_{i=1}^n x_i!}, x_i = 0, 1, \dots$$

$$\ln L(\theta) = -n\theta + n\bar{x} \ln \theta - \sum_{i=1}^n \ln x_i!$$

$$\frac{d \ln L(\theta)}{d\theta} = -n + \frac{n\bar{x}}{\theta} = 0 \Rightarrow \hat{\theta} = \bar{x}$$

$$\frac{d^2 \ln L(\theta)}{d\theta^2} \Big|_{\hat{\theta}=\bar{x}} = -n < 0$$

g) (5 pts.) Find the Fisher Information in a random variable, $I(\theta)$.

$$\lambda(x; \theta) = \ln(\theta) = -\theta + x \ln \theta - \ln x!$$

$$\lambda'(x; \theta) = -1 + \frac{x}{\theta} \Rightarrow E[\lambda'(X; \theta)] = -1 + \frac{\theta}{\theta} = 0$$

$$\lambda''(x; \theta) = -\frac{x}{\theta^2} \Rightarrow I(\theta) = -E[\lambda''(x; \theta)] = \frac{1}{\theta}$$

h) (5 pts.) Find the Cramer-Rao Lower Bound for θ .

$$CRLB = \frac{1}{\theta^2 n I(\theta)} = \frac{\theta}{n}$$

i) (5 pts.) Find the efficient estimator of θ , if it exists.

$$MLE \text{ of } \theta = \bar{X}$$

$$E(\bar{X}) = \theta \text{ and } Var(\bar{X}) = \frac{\theta}{n} = CRLB$$

Hence, \bar{X} is the EE of θ .

j) (10 pts.) Find the efficient estimator of $e^{-2\theta}$, if it exists. If it does not exist, find the asymptotic efficient estimator of $e^{-2\theta}$, and specify its asymptotic distribution.

$$MLE \text{ of } e^{2\theta} = e^{2\hat{\theta}} = e^{2\bar{X}} = e^{\frac{2}{n} \sum_{i=1}^n x_i}$$

$$X \sim Poi(\theta) \Rightarrow Y = \sum_{i=1}^n X_i \sim Poi(n\theta)$$

$$\begin{aligned} E\left[e^{\frac{2}{n}Y}\right] &= \sum_{y=1}^{\infty} e^{\frac{2}{n}y} \frac{e^{-n\theta} (n\theta)^y}{y!} = \sum_{y=0}^{\infty} \frac{e^{-n\theta} \left(e^{\frac{2}{n}} n\theta\right)^y}{y!} \\ &= \frac{e^{-n\theta}}{e^{-e^{\frac{2}{n}} n\theta}} \sum_{y=0}^{\infty} \frac{e^{-e^{\frac{2}{n}} n\theta} \left(e^{\frac{2}{n}} n\theta\right)^y}{y!} = \frac{e^{-n\theta}}{e^{-e^{\frac{2}{n}} n\theta}} = e^{-n\theta \left(1 - e^{\frac{2}{n}}\right)} \neq e^{2\theta} \end{aligned}$$

An EE of $e^{-2\theta}$ does not exist.

4. (10 pts.) Give one real life example to explain why we need the minimum variance unbiased estimators and discuss their properties using the example.

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FINAL EXAM

1. Suppose that X_1, X_2, \dots, X_n are i.i.d. with a common pdf:

$$f(x) = \begin{cases} \frac{2}{x^3} & \text{if } 1 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Find some number c such that $\bar{X}_n \xrightarrow{P} c$.

2. Let X have the p.d.f

$$f(x; \theta) = \frac{3x^2}{\theta^3}; \quad 0 < x < \theta; \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

g) (7 pts.) Find a sufficient statistic, Y for θ by using the Neyman's factorization theorem.

h) (8 pts.) Show that Y is a complete sufficient statistic for θ .

i) (10 pts.) Find the unique MVUE of $\frac{1}{\theta}$.

- PDF of $X_{(n)}$: $g_{X_{(n)}} = n f_X(x_{(n)}) [F_X(x_{(n)})]^{n-1}, -\infty < x_{(n)} < \infty$
- PDF of $X_{(1)}$: $g_{X_{(1)}} = n f_X(x_{(1)}) [1 - F_X(x_{(1)})]^{n-1}, -\infty < x_{(1)} < \infty$
- Leibnitz's Rule:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x; t) dx = f(b(t); t) \left(\frac{d}{dt} [b(t)] \right) - f(a(t); t) \left(\frac{d}{dt} [a(t)] \right) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x; t) dx$$

3. (25 pts.) The inverse Gaussian pdf

$$f(x; \theta_1, \theta_2) = \left(\frac{\theta_2}{2\pi x^3} \right)^{1/2} \exp \left\{ -\frac{\theta_2 (x - \theta_1)^2}{2\theta_1^2 x} \right\}; \quad 0 < x < \infty, \quad \theta_1 > 0 \quad \text{and} \quad \theta_2 > 0$$

is often used to model lifetimes. Find the joint complete sufficient statistics for (θ_1, θ_2) , if X_1, X_2, \dots, X_n is a random sample from the distribution having this pdf.

4. Let X_1, X_2, \dots, X_n be i.i.d. r.v.s with pdf

$$f(x; \theta) = \frac{2}{\theta} x e^{-x^2 / \theta}, x > 0, \theta > 0$$

- a) (5 pts.) Find the Maximum Likelihood Estimator (MLE) of θ .
 - b) (5 pts.) Find the Cramer-Rao Lower Bound (CRLB) for θ .
 - c) (5 pts.) Is the MLE of θ the unique Minimum Variance Unbiased Estimator (MVUE) of θ ? Explain.
 - d) (5 pts.) Find the CRLB for θ^2 .
 - e) (5 pts.) Find the efficient estimator of θ^2 , if it exists. If it does not exist, find the asymptotic efficient estimator of θ^2 , and specify its asymptotic distribution.
5. Let Y be complete sufficient statistics for the unknown parameter θ and a function of sample random variables; $u(X_1, X_2, \dots, X_n)$ is an unbiased estimator of $m(\theta)$; a function of θ .
- a) (7 pts.) How can we find the unique minimum variance unbiased estimator of $m(\theta)$ by using $u(X_1, X_2, \dots, X_n)$ and Y ? Explain.
 - b) (5 pts.) Which theorems did you use to answer the question a)?
 - c) (8 pts.) Why we are trying to find complete sufficient statistics for a parameter and a minimum variance unbiased estimator of the parameter?

a+ b)

$$E[u(X_1, X_2, \dots, X_n)] = m(\theta).$$

Let $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ be a function of Y only.

By **Rao-Blackwell Theorem**;

$$E = \{\varphi(Y)\} = E\{E[u(X_1, X_2, \dots, X_n) | Y]\} = m(\theta).$$

Hence, $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ is an UE of $m(\theta)$.

Since Y is complete sufficient statistics for θ and $u(X_1, X_2, \dots, X_n)$ is an UE of $m(\theta)$, $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ is an $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ and a function of Y . By **Lehmann-Scheffe Theorem**, $\varphi(Y) = E[u(X_1, X_2, \dots, X_n) | Y]$ is the unique MVUE of $m(\theta)$.

STAT 271/FALL09 MIDTERM EXAM I

SOLUTION

1. Suppose that X_1, X_2, \dots, X_n are i.i.d. random variables with common density

$$f(x) = \alpha x^{\alpha-1}, x \geq 1$$

where $\alpha > 0$. Define $S_n = \left(\prod_{i=1}^n X_i \right)^{1/n}$.

- a) (7 pts.) Show that $-\ln X_i$ has an Exponential distribution.

- b) (8 pts.) Show that $S_n \xrightarrow{P} \exp(-1/\alpha)$. (Hint: Consider $\ln S_n$.)

($X \sim \text{Exp}(\theta) = \text{Gamma}(1, \theta)$. $Y \sim \text{Gamma}(\alpha, \beta) \Rightarrow E(X) = \alpha\beta$ and $\text{Var}(X) = \alpha\beta^2$)

a) $Y = -\ln X \Rightarrow X = e^{-Y} \left| \frac{dx}{dy} \right| = e^{-y}$

$$f(y) = \alpha \left[e^{-\alpha y} \right]^{\alpha-1} e^{-y} = \alpha e^{-\alpha y} y > 0 \Rightarrow \text{the pdf of } \text{EXP}(1/\alpha)$$

b) $S_n = \left(\prod_{i=1}^n X_i \right)^{1/n} \Rightarrow \ln S_n = \frac{\sum_{i=1}^n \ln X_i}{n} \Rightarrow -\ln S_n = \frac{\sum_{i=1}^n -\ln X_i}{n} \sim \text{Gamma}(n, 1/n\alpha)$

i) $E(-\ln S_n) = n \frac{1}{n\alpha} = \frac{1}{\alpha}$

ii) $\text{Var}(-\ln S_n) = \frac{1}{n\alpha^2} < \infty$ for all n .

iii) $\lim_{n \rightarrow \infty} \text{Var}(-\ln S_n) = 0$

By Chebyshev's Inequality, $-\ln S_n \xrightarrow{P} 1/\alpha$.

$$\Rightarrow \ln S_n \xrightarrow{P} -1/\alpha$$

$$\Rightarrow S_n \xrightarrow{P} e^{-1/\alpha}$$

2. (20 pts.) Suppose that X_1, X_2, \dots, X_n be i.i.d. random variables with $\text{Gamma}(\alpha, n)$ p.d.f. with mean $\mu = n\alpha$ and variance $\sigma^2 = \alpha n^2$. Let Y_n be a sequence of random variables and $Y_n \xrightarrow{p} \alpha$. Find the limiting distribution of

$$\sqrt{Y_n} \cdot \left[\sqrt{\alpha} \left(\frac{\bar{X}_n}{n\alpha} - 1 \right) \right].$$

If $X \sim \text{Gamma}(\alpha, n)$, $E(\bar{X}_n) = \alpha n$, $\text{Var}(\bar{X}_n) = \frac{\alpha n^2}{n} = \alpha n$.

By CLT,

$$\frac{\bar{X}_n - \alpha n}{\sqrt{\alpha n}} = \sqrt{\alpha n} \left(\frac{\bar{X}_n}{\alpha n} - 1 \right) \xrightarrow{d} N(0, 1)$$

Since $Y_n \xrightarrow{p} \alpha$, $\sqrt{Y_n} \xrightarrow{p} \sqrt{\alpha}$. Hence,

$$\sqrt{Y_n} \left[\sqrt{\alpha n} \left(\frac{\bar{X}_n}{\alpha n} - 1 \right) \right] \xrightarrow{d} \sqrt{\alpha} N(0, 1) = N(0, \alpha).$$

3. Suppose that X_1, X_2, \dots, X_n be a random sample with $\text{Uniform}\left(\theta - \frac{1}{n}, \theta + \frac{1}{n}\right)$ p.d.f.

$$f(x) = \frac{n}{2}, \theta - \frac{1}{n} \leq x \leq \theta + \frac{1}{n}, \theta > 0.$$

- a) Let $X_{(1)} = \min(X_1, X_2, \dots, X_n)$. Find the limiting distribution of $X_{(1)}$.
b) Let $X_{(n)} = \max(X_1, X_2, \dots, X_n)$. Find the limiting distribution of $Z_n = n - n^2(X_{(n)} - \theta)$.

$$F(x) = \int_{\theta - \frac{1}{n}}^x \frac{n}{2} dx = \frac{n}{2} \left(x - \theta + \frac{1}{n} \right), \theta - \frac{1}{n} \leq x \leq \theta + \frac{1}{n}$$

a) $G_{X_{(1)}}(x_{(1)}) = 1 - [1 - F_X(x_{(1)})]^n = 1 - \left[1 - \frac{n}{2} \left(x_{(1)} - \theta + \frac{1}{n} \right) \right]^n = 1 - \frac{1}{2^n} (1 - n(x_{(1)} - \theta))^n$

$$\begin{array}{ccc} & n \rightarrow \infty & \\ \downarrow & & \downarrow \\ 0 & & \infty \\ \underbrace{\hspace{1cm}} & & \\ 0 & & \end{array}$$

$\lim_{n \rightarrow \infty} G_{X_{(1)}}(x_{(1)}) = 1, x \geq \theta, \text{ o.w. } 0$ which is the cdf of degenerate distribution at θ .

$$\text{b) } G_{X_{(n)}}(x_{(n)}) = [F_X(x_{(n)})]^n = \left[\frac{n}{2} \left(x_{(n)} - \theta + \frac{1}{n} \right) \right]^n, \theta - \frac{1}{n} \leq x_{(n)} \leq \theta + \frac{1}{n}$$

$$\begin{aligned} H_{Z_n}(z) &= P(Z_n \leq z) = P\left(n - n^2(X_{(n)} - \theta) \leq z\right) = P\left(X_{(n)} \geq \theta + \frac{n-z}{n^2}\right) = 1 - P\left(X_{(n)} \leq \theta + \frac{n-z}{n^2}\right) \\ &= 1 - \left[\frac{n}{2} \left(\theta + \frac{n-z}{n^2} - \theta + \frac{1}{n} \right) \right]^n = 1 - \left(1 - \frac{z}{2n} \right)^n, 0 \leq z \leq 2n \end{aligned}$$

$$\lim_{n \rightarrow \infty} H_{Z_n}(z) = \lim_{n \rightarrow \infty} \left\{ 1 - \left(1 - \frac{z}{2n} \right)^n \right\} = 1 - e^{-z/2}, z > 0$$

which is the cdf of Exponential(2). So, the limiting distribution of Z_n is Exp(2).

4. Let X_1, X_2, \dots, X_n be a random sample from the following discrete distribution

$$P(X_i = 1) = \frac{2\theta}{3-\theta}, P(X_i = 2) = \frac{2(1-\theta)}{3-\theta} \text{ and } P(X_i = 3) = \frac{(1-\theta)}{3-\theta}$$

where $\theta \in (0,1)$ is unknown. Find the method of moment estimator of θ .

$$E(X) = \frac{2\theta}{3-\theta} + 2 \frac{2(1-\theta)}{3-\theta} + 3 \frac{1-\theta}{3-\theta} = \frac{7-5\theta}{3-\theta}$$

To find the MME of θ , we need to equate $E(X)$ to \bar{X} .

$$\begin{aligned} E(X) &= \bar{X} \\ \frac{7-5\theta}{3-\theta} &= \bar{X} \\ \tilde{\theta} &= \frac{3\bar{X}-7}{\bar{X}} - 5 \end{aligned}$$

5. Suppose that X_1, X_2, \dots, X_n be i.i.d. random variables with Uniform(α, β) density.

Find the method of moment estimators of α and β .

Uniform

pdf:

$$f(x) = \frac{1}{\beta - \alpha}, 0 < \alpha \leq x \leq \beta$$

$$\text{with } E(X) = (\alpha + \beta)/2 \text{ and } \text{Var}(X) = \frac{(\beta - \alpha)^2}{12}.$$

$$E(X) = \bar{X}$$

$$\frac{(\alpha + \beta)}{2} = \bar{X}$$

$$(\alpha + \beta)^2 = 4\bar{X}^2$$

$$\alpha^2 + 2\alpha\beta + \beta^2 = 4\bar{X}^2$$

$$V(X) = S^2 = \text{sample variance}$$

$$\frac{(\beta - \alpha)^2}{12} = S^2$$

$$\beta^2 - 2\alpha\beta + \alpha^2 = 12S^2$$

$$\alpha^2 + 2\alpha\beta + \beta^2 = 4\bar{X}^2$$

$$\beta^2 - 2\alpha\beta + \alpha^2 = 12S^2$$

$$4\alpha\beta = 4\bar{X}^2 - 12S^2$$

$$\alpha = \frac{\bar{X}^2 - 3S^2}{\beta}$$

$$\frac{(\alpha + \beta)}{2} = \bar{X} \Rightarrow \frac{\bar{X}^2 - 3S^2}{\beta} + \beta = 2\bar{X} \Rightarrow \beta^2 - 2\bar{X}\beta + (\bar{X}^2 - 3S^2) = 0$$

$$\tilde{\beta} = \bar{X} + \sqrt{3}S$$

$$\tilde{\alpha} = \frac{\bar{X}^2 - 3S^2}{\tilde{\beta}} = \bar{X} - \sqrt{3}S$$

6. What is the reason of finding the limiting distribution of a random variable? Give an example of a case that we use the limiting distributions.

**STAT 271/FALL09
MIDTERM EXAM 2**

QUESTIONS

1. Let X have the p.d.f.

$$f(x; \theta) = \theta x^{\theta-1}, 0 < x < 1; \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

- a) (5 pts.) Find the MLE of θ .
- b) (7 pts.) Is the MLE of θ an unbiased estimator of θ ? If not, find an unbiased estimator of θ in terms of the MLE of θ .
- c) (7 pts.) Is the MLE of θ a consistent estimator of θ ?
- d) (6 pts.) Find the MLE of the median.

Hint: Think about $-\ln X$ transformation. Also, the pdf of $\text{Gamma}(\alpha, \beta)$ is:

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, x > 0, \alpha > 0, \beta > 0$$

2. Let X have the p.d.f.

$$f(x; \theta) = \frac{2x}{\theta^2}, 0 < x < \theta; \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

- j) (8 pts.) Find a sufficient statistic, Y for θ by using the Neyman's factorization theorem.
- k) (8 pts.) Show that Y is a complete sufficient statistic for θ .
- l) (9 pts.) Find the unique MVUE of θ .

- PDF of $X_{(n)}$: $g_{X_{(n)}} = n f_X(x_{(n)}) [F_X(x_{(n)})]^{n-1}, -\infty < x_{(n)} < \infty$
- PDF of $X_{(1)}$: $g_{X_{(1)}} = n f_X(x_{(1)}) [1 - F_X(x_{(1)})]^{n-1}, -\infty < x_{(1)} < \infty$
- Leibniz's Rule:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x; t) dx = f(b(t); t) \left(\frac{d}{dt} [b(t)] \right) - f(a(t); t) \left(\frac{d}{dt} [a(t)] \right) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x; t) dx$$

3. Let X have the p.d.f.

$$f(x; \theta) = \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}, 0 \leq x < \infty, 0 < \theta$$

where $\mu = \theta^{1/2} \frac{\sqrt{\pi}}{2}$, $\sigma^2 = \theta \left[1 - \frac{\pi}{4} \right]$ and $E[X^r] = \theta^{r/2} \Gamma\left(1 + \frac{r}{2}\right)$ where

$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!$, $n > 0$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Assume that we have a

random sample of size n .

a) (10 pts.) Find a complete sufficient statistics for θ .

b) (10 pts.) Find the unique minimum variance unbiased estimator of θ .

c) (10 pts.) Find an unbiased estimator of $\sqrt{\theta}$. Is this also the minimum variance unbiased estimator of $\sqrt{\theta}$? Why or why not?

d) (10 pts.) Find $E\left[\left(2X^2 + 3\right) \sum_{i=1}^n X_i^2\right]$.

4. (10 pts.) Explain the logic behind the maximum likelihood estimation. What is a likelihood function and why we are trying to maximize the likelihood function?

STAT 271/09 FINAL EXAM

QUESTIONS

1. (20 pts.) Let $F(x)/x \rightarrow \lambda$ as $x \rightarrow 0$ where $F(x)$ is a cumulative distribution function of X . If $X_{(1)}$ is the first order statistics for a r.s. of size n , find the limiting distribution of $X_{(1)}$, if it exists.

$$\text{cdf of } X_{(1)}: G_{X_{(1)}}(x_{(1)}) = 1 - [1 - F_X(x_{(1)})]^n, -\infty < x_{(1)} < \infty.$$

2. Consider a r.s. X_1, X_2, \dots, X_n from the distribution having the following pdf

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}; 0 < x < \infty, \alpha > 0 \text{ and } \beta > 0$$

with $E(X) = \alpha\beta$ and $Var(X) = \alpha\beta^2$.

- a) (10 pts.) Find the joint complete sufficient statistics for (α, β) .
- b) (15 pts.) If $\alpha=1$; find the MVUE of $1/\beta^2$.

3. Let X_1, X_2, \dots, X_n be i.i.d. r.v.s with pdf

$$f(x; \theta) = \frac{1}{\theta} x^{-\left[\frac{1-\theta}{\theta}\right]}, 0 < x < 1; \theta > 0$$

- a) (5 pts.) Find the maximum likelihood estimator (MLE) of θ .
- b) (5 pts.) Find the Fisher Information, $I(\theta)$.
- c) (5 pts.) Find the Cramer-Rao Lower Bound (CRLB) for θ .
- d) (5 pts.) Find the efficient estimator of θ , if it exists.
- e) (10 pts.) Find the efficient estimator of θ^2 , if it exists. If it does not exist, find the asymptotic efficient estimator of θ^2 , and specify its asymptotic distribution.

4.

- a) (5 pts.) What is a random variable? Why we need them in statistical analysis?
- b) (5 pts.) What is a random sample? Why we want to have a random sample?
- c) (5 pts.) Why we are defining probability mass function for discrete random variables and probability density function for continuous random variables? Are they the same? If not, what are the differences?
- d) (5 pts.) What is the meaning of Cramer Rao Lower Bound? Why we need them?
- e) (5 pts.) What is the meaning of the efficient estimator?

STAT 271
MIDTERM EXAM I

SOLUTION

7. Suppose that X_1, X_2, \dots, X_n are i.i.d. random variables with common density

$$f(x) = \alpha x^{\alpha-1}, x \geq 1$$

where $\alpha > 0$. Define $S_n = \left(\prod_{i=1}^n X_i \right)^{1/n}$.

c) (7 pts.) Show that $-\ln X_i$ has an Exponential distribution.

d) (8 pts.) Show that $S_n \xrightarrow{P} \exp(-1/\alpha)$. (Hint: Consider $\ln S_n$.)

($X \sim \text{Exp}(\theta) = \text{Gamma}(1, \theta)$. $Y \sim \text{Gamma}(\alpha, \beta) \Rightarrow E(X) = \alpha\beta$ and $\text{Var}(X) = \alpha\beta^2$)

c) $Y = -\ln X \Rightarrow X = e^{-Y} \left| \frac{dx}{dy} \right| = e^{-y}$

$$f(y) = \alpha \left[e^{-\alpha y} \right]^{\alpha-1} e^{-y} = \alpha e^{-\alpha y} y > 0 \Rightarrow \text{the pdf of } \text{EXP}(1/\alpha)$$

d)

8. (20 pts.) Suppose that X_1, X_2, \dots, X_n be i.i.d. random variables with $\text{Gamma}(\alpha, n)$ p.d.f. with mean $\mu = n\alpha$ and variance $\sigma^2 = \alpha n^2$. Let Y_n be a sequence of random variables and $Y_n \xrightarrow{P} \alpha$. Find the limiting distribution of

$$\sqrt{Y_n} \cdot \left[\sqrt{\alpha} \left(\frac{\bar{X}_n}{n\alpha} - 1 \right) \right].$$

If $X \sim \text{Gamma}(\alpha, n)$, $E(\bar{X}_n) = \alpha n$, $\text{Var}(\bar{X}_n) = \frac{\alpha n^2}{n} = \alpha n$.

By CLT,

$$\frac{\bar{X}_n - \alpha n}{\sqrt{\alpha n}} = \sqrt{\alpha n} \left(\frac{\bar{X}_n}{\alpha n} - 1 \right) \xrightarrow{d} N(0,1)$$

Since $Y_n \xrightarrow{P} \alpha$, $\sqrt{Y_n} \xrightarrow{P} \sqrt{\alpha}$. Hence,

$$\sqrt{Y_n} \left[\sqrt{\alpha n} \left(\frac{\bar{X}_n}{\alpha n} - 1 \right) \right] \xrightarrow{d} \sqrt{\alpha} N(0,1) = N(0, \alpha).$$

9. Suppose that X_1, X_2, \dots, X_n be a random sample with $Uniform\left(\theta - \frac{1}{n}, \theta + \frac{1}{n}\right)$ p.d.f.

$$f(x) = \frac{n}{2}, \theta - \frac{1}{n} \leq x \leq \theta + \frac{1}{n}, \theta > 0.$$

- c) Let $X_{(1)} = \min(X_1, X_2, \dots, X_n)$. Find the limiting distribution of $X_{(1)}$.

- d) Let $X_{(n)} = \max(X_1, X_2, \dots, X_n)$. Find the limiting distribution of $Z_n = n - n^2(X_{(n)} - \theta)$.

$$F(x) = \int_{\theta - \frac{1}{2}}^x \frac{n}{2} dx = \frac{n}{2} \left(x - \theta + \frac{1}{n} \right), \theta - \frac{1}{n} \leq x \leq \theta + \frac{1}{n}$$

$$c) G_{X_{(1)}}(x_{(1)}) = 1 - [1 - F_X(x_{(1)})]^n = 1 - \left[1 - \frac{n}{2} \left(x_{(1)} - \theta + \frac{1}{n} \right) \right]^n = 1 - \frac{1}{2^n} (1 - n(x_{(1)} - \theta))^n$$

$$\begin{array}{ccc} \downarrow & n \rightarrow \infty & \downarrow \\ 0 & & \infty \\ \underbrace{\hspace{1.5cm}} & & \\ 0 & & \end{array}$$

$\lim_{n \rightarrow \infty} G_{X_{(1)}}(x_{(1)}) = 1, x \geq \theta, o.w. 0$ which is the cdf of degenerate distribution at θ .

$$d) G_{X_{(n)}}(x_{(n)}) = [F_X(x_{(n)})]^n = \left[\frac{n}{2} \left(x_{(n)} - \theta + \frac{1}{n} \right) \right]^n, \theta - \frac{1}{n} \leq x_{(n)} \leq \theta + \frac{1}{n}$$

$$\begin{aligned} H_{Z_n}(z) &= P(Z_n \leq z) = P\left(n - n^2(X_{(n)} - \theta) \leq z\right) = P\left(X_{(n)} \geq \theta + \frac{n-z}{n^2}\right) = 1 - P\left(X_{(n)} \leq \theta + \frac{n-z}{n^2}\right) \\ &= 1 - \left[\frac{n}{2} \left(\theta + \frac{n-z}{n^2} - \theta + \frac{1}{n} \right) \right]^n = 1 - \left(1 - \frac{z}{2n} \right)^n, 0 \leq z \leq 2n \end{aligned}$$

$$\lim_{n \rightarrow \infty} H_{Z_n}(z) = \lim_{n \rightarrow \infty} \left\{ 1 - \left(1 - \frac{z}{2n} \right)^n \right\} = 1 - e^{-z/2}, z > 0$$

which is the cdf of Exponential(2). So, the limiting distribution of Z_n is Exp(2).

10. Let X_1, X_2, \dots, X_n be a random sample from the following discrete distribution

$$P(X_i = 1) = \frac{2\theta}{3-\theta}, P(X_i = 2) = \frac{2(1-\theta)}{3-\theta} \text{ and } P(X_i = 3) = \frac{(1-\theta)}{3-\theta}$$

where $\theta \in (0,1)$ is unknown. Find the method of moment estimator of θ .

$$E(X) = \frac{2\theta}{3-\theta} + 2 \frac{2(1-\theta)}{3-\theta} + 3 \frac{1-\theta}{3-\theta} = \frac{7-5\theta}{3-\theta}$$

To find the MME of θ , we need to equate $E(X)$ to \bar{X} .

$$\begin{aligned} E(X) &= \bar{X} \\ \frac{7-5\theta}{3-\theta} &= \bar{X} \\ \tilde{\theta} &= \frac{3\bar{X}-7}{\bar{X}} - 5 \end{aligned}$$

11. Suppose that X_1, X_2, \dots, X_n be i.i.d. random variables with Uniform(α, β) density. Find the method of moment estimators of α and β .

Uniform pdf: $f(x) = \frac{1}{\beta-\alpha}, 0 < \alpha \leq x \leq \beta$

with $E(X) = (\alpha + \beta)/2$ and $Var(X) = \frac{(\beta - \alpha)^2}{12}$.

$$\begin{aligned} E(X) &= \bar{X} \\ \frac{(\alpha + \beta)}{2} &= \bar{X} \\ (\alpha + \beta)^2 &= 4\bar{X}^2 \\ \alpha^2 + 2\alpha\beta + \beta^2 &= 4\bar{X}^2 \end{aligned}$$

$$V(X) = S^2 = \text{sample variance}$$

$$\frac{(\beta - \alpha)^2}{12} = S^2$$

$$\beta^2 - 2\alpha\beta + \alpha^2 = 12S^2$$

$$\alpha^2 + 2\alpha\beta + \beta^2 = 4\bar{X}^2$$

$$\beta^2 - 2\alpha\beta + \alpha^2 = 12S^2$$

$$4\alpha\beta = 4\bar{X}^2 - 12S^2$$

$$\alpha = \frac{\bar{X}^2 - 3S^2}{\beta}$$

$$\frac{(\alpha + \beta)}{2} = \bar{X} \Rightarrow \frac{\bar{X}^2 - 3S^2}{\beta} + \beta = 2\bar{X} \Rightarrow \beta^2 - 2\bar{X}\beta + (\bar{X}^2 - 3S^2) = 0$$

$$\tilde{\beta} = \bar{X} + \sqrt{3}S$$

$$\tilde{\alpha} = \frac{\bar{X}^2 - 3S^2}{\tilde{\beta}} = \bar{X} - \sqrt{3}S$$

12. What is the reason of finding the limiting distribution of a random variable? Give an example of a case that we use the limiting distributions.

Dr. Ceylan Yozgatligil
METU-FALL 2009

21.12.2009

STAT 271
MIDTERM EXAM 2

QUESTIONS

6. Let X have the p.d.f.

$$f(x; \theta) = \theta x^{\theta-1}, 0 < x < 1; \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

- e) (5 pts.) Find the MLE of θ .
- f) (7 pts.) Is the MLE of θ an unbiased estimator of θ ? If not, find an unbiased estimator of θ in terms of the MLE of θ .
- g) (7 pts.) Is the MLE of θ a consistent estimator of θ ?
- h) (6 pts.) Find the MLE of the median.

Hint: Think about $-\ln X$ transformation. Also, the pdf of $\text{Gamma}(\alpha, \beta)$ is:

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, x > 0, \alpha > 0, \beta > 0$$

7. Let X have the p.d.f.

$$f(x; \theta) = \frac{2x}{\theta^2}, 0 < x < \theta; \theta > 0$$

Let X_1, \dots, X_n be a random sample of size n .

m) (8 pts.) Find a sufficient statistic, Y for θ by using the Neyman's factorization theorem.

n) (8 pts.) Show that Y is a complete sufficient statistic for θ .

o) (9 pts.) Find the unique MVUE of θ .

- PDF of $X_{(n)}$: $g_{X_{(n)}} = n f_X(x_{(n)}) [F_X(x_{(n)})]^{n-1}, -\infty < x_{(n)} < \infty$
- PDF of $X_{(1)}$: $g_{X_{(1)}} = n f_X(x_{(1)}) [1 - F_X(x_{(1)})]^{n-1}, -\infty < x_{(1)} < \infty$
- Leibniz's Rule:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x; t) dx = f(b(t); t) \left(\frac{d}{dt} [b(t)] \right) - f(a(t); t) \left(\frac{d}{dt} [a(t)] \right) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x; t) dx$$

8. Let X have the p.d.f.

$$f(x; \theta) = \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}, 0 \leq x < \infty, 0 < \theta$$

where $\mu = \theta^{1/2} \frac{\sqrt{\pi}}{2}$, $\sigma^2 = \theta \left[1 - \frac{\pi}{4} \right]$ and $E[X^r] = \theta^{r/2} \Gamma\left(1 + \frac{r}{2}\right)$ where

$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!$, $n > 0$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Assume that we have a random sample of size n .

a) (10 pts.) Find a complete sufficient statistics for θ .

b) (10 pts.) Find the unique minimum variance unbiased estimator of θ .

c) (10 pts.) Find an unbiased estimator of $\sqrt{\theta}$. Is this also the minimum variance unbiased estimator of $\sqrt{\theta}$? Why or why not?

d) (10 pts.) Find $E\left[\left(2X^2 + 3\right)\sum_{i=1}^n X_i^2\right]$.

9. (10 pts.) Explain the logic behind the maximum likelihood estimation. What is a likelihood function and why we are trying to maximize the likelihood function?

Dr. Ceylan Yozgatligil
METU-FALL 2009

18.01.2010

**STAT 271
FINAL EXAM
QUESTIONS**

2. (20 pts.) Let $F(x)/x \rightarrow \lambda$ as $x \rightarrow 0$ where $F(x)$ is a cumulative distribution function of X . If $X_{(1)}$ is the first order statistics for a r.s. of size n , find the limiting distribution of $X_{(1)}$, if it exists.

$$\text{cdf of } X_{(1)}: G_{X_{(1)}}(x_{(1)}) = 1 - [1 - F_X(x_{(1)})]^n, -\infty < x_{(1)} < \infty.$$

2. Consider a r.s. X_1, X_2, \dots, X_n from the distribution having the following pdf

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}; \quad 0 < x < \infty, \quad \alpha > 0 \quad \text{and} \quad \beta > 0$$

with $E(X) = \alpha\beta$ and $\text{Var}(X) = \alpha\beta^2$.

f) (10 pts.) Find the joint complete sufficient statistics for (α, β) .

g) (15 pts.) If $\alpha=1$; find the MVUE of $1/\beta^2$.

3. Let X_1, X_2, \dots, X_n be i.i.d. r.v.s with pdf

$$f(x; \theta) = \frac{1}{\theta} x^{-\left[\frac{1-\theta}{\theta}\right]}; \quad 0 < x < 1; \theta > 0$$

- f) (5pts.) Find the maximum likelihood estimator (MLE) of θ ,
- g) (5pts.) Find the Fisher Information, $I(\theta)$.
- h) (5 pts.) Find the Cramer-Rao Lower Bound (CRLB) for θ .
- i) (5 pts.) Find the efficient estimator of θ , if it exists.
- j) (10 pts.) Find the efficient estimator of θ^2 , if it exists. If it does not exist, find the asymptotic efficient estimator of θ^2 , and specify its asymptotic distribution.

4.

- b) (5pts.) What is a random variable? Why we need them in statistical analysis?
- b) (5pts.) What is a random sample? Why we want to have a random sample?
- h) (5pts.) Why we are defining probability mass function for discrete random variables and probability density function for continuous random variables? Are they the same? If not, what are the differences?
- i) (5pts.) What is the meaning of Cramer Rao Lower Bound? Why we need them?
- j) (5pts.) What is the meaning of the efficient estimator?

Dr. Ceylan Yozgatligil
14.11.2011
METU-FALL 2011

STAT 271 MIDTERM EXAM I

SOLUTIONS

- Let $X = (X_1, \dots, X_n)$ be a random sample from the density

$$f(x) = \frac{3}{x^4}, x \geq 1.$$

- a. (5 pts.) Find the limiting distribution of Y_L , if exists.

Step 1: Find the cdf of X . $F(x) = 1 - x^{-3}, x \geq 1$

Step 2: Find the cdf of Y_L : $G_{Y_L}(y) = 1 - [1 - 1 + y^{-3}]^n = 1 - y^{-3n}, y \geq 1$

Step 3: Take the limit of the cdf: $\lim_{n \rightarrow \infty} G_{Y_1}(y) = \lim_{n \rightarrow \infty} (1 - y^{-3n}) = \begin{cases} 1, y \geq 1 \\ 0, y < 1 \end{cases}$

which is the cdf of a degenerate distribution at point 1. Hence, the limiting distribution of Y_1 is degenerate distribution at point $y=1$.

b. (5 pts.) Find the limiting distribution of Y_n , if exists.

Step 1: Find the cdf of Y_n : $G_{Y_n}(y) = [1 - y^{-3}]^n$

Step 2: Take the limit of the cdf:

$$\lim_{n \rightarrow \infty} G_{Y_n}(y) = \lim_{n \rightarrow \infty} (1 - y^{-3})^n = \begin{cases} 0, y \geq 1 \\ 0, y < 1 \end{cases}$$

which is not a valid cdf. Hence, the limiting distribution of Y_n does not exist.

c. (10 pts.) Find the limiting distribution of $Z_n = \sqrt[3]{n} Y_n$, if exists.

Step 1: Find the cdf of Z_n :

$$H_{Z_n}(z) = P(Z_n \leq z) = P(\sqrt[3]{n} Y_n \leq z) = P(Y_n \leq z / \sqrt[3]{n}) = \left(1 - \frac{n}{z^3}\right)^n, z / \sqrt[3]{n} \geq 1$$

Step 2: Take the limit of the cdf:

$$\lim_{n \rightarrow \infty} H_{Z_n}(y) = \lim_{n \rightarrow \infty} \left(1 - \frac{n}{z^3}\right)^n = -\infty^\infty, z \geq \infty$$

which is not a valid cdf. Hence, the limiting distribution of Y_n does not exist.

• (15 pts.) Let X_1, \dots, X_n be a random sample from the $Uniform(0,1)$ density. Let

$$Z_n = \left(\prod_{i=1}^n X_i \right)^{1/n}. \text{ Show that } Z_n \xrightarrow{p} c, \text{ where } c \text{ is constant and find } c?$$

Here, the case is stochastic convergence. So, we need to find the $E(Z_n)$ and $V(Z_n)$. To be able to do so, we need the distribution of Z_n . It is not easy to find the distribution of Z_n , therefore consider a natural logarithm transformation to make the form of Z_n linear.

$$Y_n = \ln Z_n = \frac{1}{n} \sum_{i=1}^n \ln X_i \Rightarrow \text{Consider } -\ln X \text{ transformation to find the distribution of } Y_n.$$

$$U = -\ln X \Rightarrow X = e^{-U} \Rightarrow \text{Jacobian} = \left| \frac{de^{-u}}{du} \right| = e^{-u} \Rightarrow f(u) = e^{-u}, u > 0 \Rightarrow U \sim \text{Exp}(1) = \text{Gamma}(1,1).$$

Hence, $\sum_{i=1}^n U_i = -\sum_{i=1}^n \ln X_i \sim \text{Gamma}(n,1) \Rightarrow -\frac{1}{n} \sum_{i=1}^n \ln X_i \sim \text{Gamma}\left(n, \frac{1}{n}\right)$. Then,

$$E\left[-\frac{\sum_{i=1}^n \ln X_i}{n}\right] = 1 \text{ and } V\left[-\frac{\sum_{i=1}^n \ln X_i}{n}\right] = \frac{1}{n}$$

By Chebyshev's Inequality,

$$i) E\left[-\frac{\sum_{i=1}^n \ln X_i}{n}\right] = 1$$

$$ii) V\left[-\frac{\sum_{i=1}^n \ln X_i}{n}\right] = \frac{1}{n} < \infty, \forall n$$

$$iii) \lim_{n \rightarrow \infty} V\left[-\frac{\sum_{i=1}^n \ln X_i}{n}\right] = 0$$

$$\text{Hence, } -\frac{\sum_{i=1}^n \ln X_i}{n} \xrightarrow{p} 1.$$

$$-\exp\left\{\frac{\sum_{i=1}^n \ln X_i}{n}\right\} \xrightarrow{p} \exp\{1\}$$

$$\left(\prod_{i=1}^n X_i\right)^{1/n} \xrightarrow{p} e^{-1}.$$

- (15 pts.) Suppose that $Y_i \sim N(0,1)$ and Y_1, Y_2, \dots are independent. Use mgf method to

find the limiting distribution of $\frac{\sum_{i=1}^n \left(Y_i + \frac{1}{n} \right)}{\sqrt{n}}$.

$$\text{Let } Z_n = \frac{\sum_{i=1}^n \left(Y_i + \frac{1}{n} \right)}{\sqrt{n}} = \frac{\sum_{i=1}^n Y_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}.$$

Mgf

of

Z_n :

$$M_{Z_n}(t) = E \left[\exp \left(\frac{t \sum_{i=1}^n Y_i}{\sqrt{n}} + \frac{t}{\sqrt{n}} \right) \right] = e^{t/\sqrt{n}} \left[M_{Y_i} \left(t/\sqrt{n} \right) \right]^n = e^{t/\sqrt{n}} \left[e^{t^2/2n} \right]^n = e^{t/\sqrt{n}} e^{t^2/2}.$$

Take the limit of the mgf: $\lim_{n \rightarrow \infty} M_{Z_n}(t) = \lim_{n \rightarrow \infty} e^{t/\sqrt{n}} e^{t^2/2} = e^{t^2/2}, -\infty < t < \infty$ which

is the mgf of $N(0,1)$. Hence, the limiting distribution of Z_n is $N(0,1)$.

- (15 pts.) Let X have the $Uniform(n, n+2)$ p.d.f. Considering X_1, \dots, X_n is a random sample of size n , find the limiting distribution of

$$Z_n = \frac{3n(\bar{X}_n - n - 1)}{\sqrt{\bar{X}_n}}$$

where $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$, if it exists.

Since $X \sim Uniform(n, n+2)$, $E(X) = \frac{n+n+2}{2} = n+1$ and $V(X) = \frac{(n+2-n)^2}{12} = \frac{1}{3}$.

By CLT, $\frac{\bar{X}_n - n - 1}{\left(\frac{1}{\sqrt{3n}}\right)} = \sqrt{3n}(\bar{X}_n - n - 1) \xrightarrow{d} N(0,1)$.

After this part, we have $\frac{\sqrt{3n}}{\sqrt{\bar{X}_n}}$.

By WLLN, $\bar{X}_n \xrightarrow{p} (n+1)$.

By Slutsky's Theorem, $\frac{\bar{X}_n}{n} \xrightarrow{p} 1$ and $\sqrt{\frac{\bar{X}_n}{n}} \xrightarrow{p} 1$.

Hence,

$$Z_n = \sqrt{3} \frac{\sqrt{3n}(\bar{X}_n - n - 1)}{\sqrt{\frac{\bar{X}_n}{n}}} \xrightarrow{d} \sqrt{3}N(0,1) = N(0,3).$$

- Let X_1, \dots, X_n be a random sample from the following density

$$f(x; \theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0.$$

- a. (10 pts.) Find the method of moment estimator (MME) of θ .

$$E(X) = \bar{X}$$

$$\int_0^1 x \theta x^{\theta-1} dx = \frac{\theta}{\theta+1} = \bar{X}$$

$$\Rightarrow \tilde{\theta} = \frac{\bar{X}}{1 - \bar{X}}$$

- b. (10 pts.) Find the maximum likelihood estimator (MLE) of θ .

$$L(\theta) = \theta^n \prod_{i=1}^n x_i^{\theta-1}, 0 < x_i < 1$$

$$\ln L(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^n \ln x_i$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln x_i = 0 \text{ and check } \left. \frac{d^2 \ln L(\theta)}{d\theta^2} \right|_{\theta=\hat{\theta}} < 0.$$

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln x_i}$$

- c. (5 pts.) As you can see from parts a) and b), the MME and MLE of θ are different than each other as the estimators of the same parameter. Discuss how we can decide to choose just one estimator among MME and MLE of θ .
- d. (10 pts.) Discuss the logic behind MME and MLE techniques.

Dr. Ceylan Yozgathgil
METU-FALL 2011

19.12.2011

STAT 271 MIDTERM EXAM II

QUESTIONS

1. (7 pts.) Let X_1, \dots, X_n be a r.s. from an $N(\mu_1, \sigma^2)$ distribution and let Y_1, \dots, Y_m be a r.s. from an $N(\mu_2, \sigma^2)$ distribution. Is the pooled sample variance

$$\hat{\sigma}_{pooled}^2 = \frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2}$$

an unbiased estimator of σ^2 , where S_1^2 and S_2^2 are the respective sample variances. Support your answer with calculations.

2. Let X_1, \dots, X_n be a random sample from a Rayleigh distribution with pdf

$$f(x; \alpha) = \begin{cases} \frac{x}{\alpha} e^{-x^2/2\alpha}, & x > 0, \alpha > 0 \\ 0 & , \text{otherwise} \end{cases}$$

- a) (5 pts.) Find the Maximum Likelihood Estimator (MLE) of α , if it exists.
b) (8 pts.) Is the MLE you found in part a) an unbiased estimator of α . Show.
c) (5 pts.) Is the MLE you found in part a) a consistent estimator of α . Show.
d) (7 pts.) Find the (MLE) of $MSE(\hat{\alpha})$, if it exists.
e) (8 pts.) Find a complete and sufficient statistic for α , if it exists.
f) (7 pts.) Find the unique minimum variance unbiased estimator of α^2 , if it exists.
3. A random sample X_1, X_2, \dots, X_n is drawn from the following pdf

$$f(x; \theta) = \frac{1}{2\theta}; \quad 3 - 2\theta \leq x \leq 3, \theta > 0$$

<ul style="list-style-type: none"> • cdf of Y_n: $G_{Y_n} = [F_X(y_n)]^n, -\infty < y_n < \infty$ • cdf of Y_1: $G_{Y_1} = 1 - [1 - F_X(y_1)]^n, -\infty < y_1 < \infty$
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- p) (8 pts.) Find a sufficient statistic, Y for θ by using the Neyman's factorization theorem.
- q) (9 pts.) Show that Y is a complete sufficient statistic for θ .
- r) (9 pts.) Find the unique MVUE of θ .
- s) (7 pts.) Find $E[3\bar{X} | Y_1]$ where \bar{X} is the sample mean and Y_1 is the first order statistic.

4. Suppose that the random variables Y_1, Y_2, \dots, Y_n satisfy

$$Y_i = \beta x_i + \varepsilon_i, \quad i = 1, 2, \dots, n$$

where x_1, x_2, \dots, x_n are fixed constants, and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are i.i.d. $N(0, \sigma^2)$, σ^2 unknown.

- a) (5 pts.) Find the pdf of Y .
- b) (10 pts.) Find joint complete sufficient statistics for (β, σ^2) .
- c) (5 pts.) Find the minimum variance unbiased estimator of β .

Dr. Ceylan Yozgathgil
METU-FALL 2011

20.01.2012

STAT 271
FINAL EXAM
Duration: 120 minutes

QUESTIONS

1. Let $X \sim N(\mu, \sigma^2)$ where μ is unknown. A random sample of size n is taken.

- (5 pts.) Assuming that μ is unknown, show that $S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} \xrightarrow{p} \sigma^2$, where \bar{X} is the sample mean.

- (5 pts.) Assuming that μ is known, show that $S'^2 = \frac{\sum_{i=1}^n (X_i - \mu)^2}{n} \xrightarrow{p} \sigma^2$.

2. The independent rvs X_1, X_2, \dots, X_n have the common distribution

$$P(X_i \leq x; \alpha, \beta) = \begin{cases} 0 & , \text{ if } x < 0 \\ \left(\frac{x}{\beta}\right)^\alpha & , \text{ if } 0 \leq x < \beta \\ 1 & , \text{ if } x \geq \beta \end{cases}$$

where α and β are positive.

- (5 pts.) Find the MLEs of α and β .
- (5 pts.) Find the MLE of interquartile range.
- (5 pts.) Find the minimal sufficient statistics of α and β .

3. Let X be a random variable having pdf

$$f(x; \theta) = \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}, 0 \leq x < \infty, 0 < \theta$$

where $\mu = \theta^{1/2} \frac{\sqrt{\pi}}{2}$, $\sigma^2 = \theta^{1/2} \left[1 - \frac{\pi}{4}\right]$ and $E[X^r] = \theta^{r/2} \Gamma\left(1 + \frac{r}{2}\right)$ where $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!$, $n > 0$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Consider a random sample, X_1, X_2, \dots, X_n .

- (5 pts.) Find the Fisher Information in a random variable, $I(\theta)$.
- (5 pts.) Find the Cramer Rao Lower Bound for θ .
- (10 pts.) Find the efficient estimator of θ , if it exists.
(Hint: You may consider $Y = X^2$ transformation, if you have difficulty in calculations.)
- (10 pts.) Find the efficient estimator of θ^2 , if it exists. If it does not exist, find the asymptotically efficient estimator of θ^2 and specify its asymptotic distribution.
(Hint: You may consider $Y = X^2$ transformation, if you have difficulty in calculations.)

4. Suppose that the random variables Y_1, Y_2, \dots, Y_n satisfy

$$Y_i = \alpha + \beta x_i + \varepsilon_i, \quad i = 1, 2, \dots, n$$

where x_1, x_2, \dots, x_n are fixed constants, and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are i.i.d. $N(0, \sigma^2)$, σ^2 unknown.

- a) (5 pts.) Find the pdf of Y .
 - b) (10 pts.) Find joint complete sufficient statistics for $(\alpha, \beta, \sigma^2)$.
 - c) (10 pts.) Find the minimum variance unbiased estimator of β .
- 5.
- a) (5pts.) What is a random variable? Why we need them in statistical analysis?
 - b) (5pts.) What is a random sample? Why we want to have a random sample?
 - c) (5pts.) What is the meaning of Cramer Rao Lower Bound? Why we need them?
 - d) (5pts.) What is the meaning of the efficient estimator?

Dr. Ceylan Yozgatligil

16.11.2017

METU-FALL 2017

STAT 303
MIDTERM EXAM I

QUESTIONS

1. If X_1, X_2, \dots, X_n , a random sample from $N(\mu_X, \sigma^2)$ and Y_1, Y_2, \dots, Y_n , a from $N(0, \sigma^2)$ are independent, find the distribution of the following random variables without any derivation:
- a) (5 pts.) $\bar{X} - \bar{Y}$
- b) (5 pts.) $S_X^2 + S_Y^2$
- c) (5 pts.) $\frac{\bar{Y}}{\sqrt{S_X^2 / n}}$
- d) (5 pts.) $\frac{(\bar{X} - \mu_X)^2}{\bar{Y}^2}$
2. (20 pts.) One observation is taken on a discrete random variable X with the following probability mass function $f(x; \theta)$, where $\theta = 1, 2$.

x	0	1	2
$f(x; \theta)$	$\frac{1}{2\theta}$	$\frac{\theta - 1}{3\theta}$	$\frac{2\theta - 1}{6\theta}$

Compute an estimate for θ using the method of moments and the maximum likelihood method, and compare them.

3. Suppose X_1, X_2, \dots, X_n is a random sample from the Uniform distribution on $-\sqrt{\theta} \leq X \leq \sqrt{\theta}$.
- a) (10 pts.) Find the maximum likelihood estimator (MLE) of θ .
- b) (7 pts.) Find the MLE of $P(X \leq 2)$.
- c) (10 pts.) Find an unbiased estimator of θ as a function of the MLE of θ .

Note:

$$g_{X_{(1)}}(x_{(1)}) = nf(x_{(1)})[1 - F(x_{(1)})]^{n-1}, -\infty < x_{(1)} < \infty$$

$$g_{x_{(n)}}\big(x_{(n)}\big)=nf\big(x_{(n)}\big)\big[F\big(x_{(n)}\big)\big]^{n-1},-\infty < x_{(n)} < \infty$$

$$g_{x_{(1)},x_{(n)}}\big(x_{(1)},x_{(n)}\big)=n(n-1)f\big(x_{(1)}\big)f\big(x_{(n)}\big)\big[F\big(x_{(n)}\big)-F\big(x_{(1)}\big)\big]^{n-2},-\infty < x_{(1)} < x_{(n)} < \infty$$

4. (8 pts.) A random variable Y has a probability density function

$$f(y) = (1 - \theta) + 2\theta y, \quad 0 < y < 1, \quad -1 < \theta < 1,$$

0 otherwise. There are n observations y_i ; $i = 1, \dots, n$, drawn independently from this distribution. Find the maximum likelihood estimator of θ . Discuss the ways of finding a maximum likelihood estimate for θ .

5. Please answer the following questions:

- a) (10 pts.) What do we mean by “sampling distribution” of an estimator $\hat{\theta}$? In particular, if $\hat{\theta}$ is unbiased, then what does this say about its sampling distribution?
- b) (15 pts.) Write a real life example to explain the likelihood concept. How do we interpret the likelihood function, $L(\theta)$? What does the maximum likelihood estimator give us?

Dr. Ceylan Yozgatligil

20.12.2017

METU-FALL 2017

STAT 303

MIDTERM EXAM II

(Time Duration: 90 minutes)

QUESTIONS

5. Let X_1, \dots, X_n be a random sample from a $\text{Poi}(\theta)$ distribution.
- a) (8 pts.) Find a sufficient statistic, Y for θ .
 - b) (7 pts.) Find an unbiased estimator of θ in terms of Y .
 - c) (8 pts.) Is the estimator that you found in part b) a consistent estimator of θ ? Show.
6. Let X_1, \dots, X_n be a random sample from a $\text{Uniform}(\theta, 2)$ distribution where $\theta > 0$.
- g) (10 pts.) Find a sufficient statistic for θ , if it exists.
 - h) (10 pts.) Find a complete sufficient statistic for θ , if it exists.
 - i) (10 pts.) Find the unique minimum variance unbiased estimator of θ , if it exists.

Hint: $g(x_{(1)}) = nf(x_{(1)})[1 - F(x_{(1)})]^{n-1}, -\infty < x_{(1)} < \infty$

$$g(x_{(n)}) = nf(x_{(n)})[F(x_{(n)})]^{n-1}, -\infty < x_{(n)} < \infty$$

7. A random sample X_1, X_2, \dots, X_n is drawn from the Gamma(2, θ) distribution where $\theta > 0$.

t) (10 pts.) Find a complete sufficient statistic, Y for θ , if it exists.

u) (12 pts.) Find the unique minimum variance unbiased estimator of θ^2 , if it exists.

v) (15 pts.) Find the unique minimum variance unbiased estimator of $1/\theta$, if it exists.

8. (10 pts) The Lehmann-Scheffe Theorem states that if a statistic Y is a complete sufficient statistic for the unknown parameter θ , then the unbiased estimator of θ as a function of Y is the unique minimum variance unbiased estimator of θ . If we cannot find an unbiased estimator of θ in terms of Y , can we still find the unique minimum variance unbiased estimator? If yes, please explain how?

Dr. Ceylan Yozgathgil
METU-FALL 2017

16.01.2018

STAT 303 **FINAL EXAM**

QUESTIONS

1. (15 pts.) If X_1, X_2, \dots, X_n , a random sample from $N(\mu_X, \sigma^2)$ and Y_1, Y_2, \dots, Y_n , a random sample from $N(0, \sigma^2)$ are independent, find the distribution of $S_X^2 + S_Y^2$, where S_X^2 and S_Y^2 are the sample variances, without any derivation.
2. The independent and identically distributed rvs X_1, X_2, \dots, X_n have the common distribution

$$f(x; \alpha, \beta) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}, x \geq \beta$$

where α and β are positive.

- (10 pts.) Find the MLEs of α and β .
 - (10 pts.) Find the joint minimal sufficient statistics of α and β .
3. Let (X_1, \dots, X_n) be independent identically distributed random variables with p.d.f.

$$f(x; \theta) = \theta^2 x \exp(-\theta x), x > 0$$

- a) (10 pts.) Find a complete sufficient statistic for θ .
- b) (10 pts.) Find the unique minimum variance unbiased estimator of θ , if it exists.
4. We consider two continuous independent random variables U and W normally distributed with $N(0, \sigma^2)$. The variable X defined by

$$X = \sqrt{U^2 + W^2}$$

has a Rayleigh distribution with p.d.f.

$$f(x; \sigma^2) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), x > 0$$

- a) (5 pts.) Find the MLEs of σ^2 .
- b) (7 pts.) Find the MLE of $P(X \leq 2)$.
- c) (7 pts.) Find the Fisher Information on a random sample.
- d) (8 pts.) Find the Cramer Rao Lower Bound for σ^2 .
- e) (8 pts.) Find the efficient estimator of σ^2 , if it exists.
(**Hint:** You may consider $Y=X^2$ transformation, if you have difficulty in calculations.)
- f) (10 pts.) Find the efficient estimator of σ^4 , if it exists. If it does not exist, find the asymptotically efficient estimator of σ^4 and specify its asymptotic distribution. (**Hint:** You may consider $Y=X^2$ transformation, if you have difficulty in calculations.)

Dr. Ceylan Yozgatligil
METU-FALL 2018

12.11.2018

STAT 303 **FINAL EXAM**

QUESTIONS

1. If random samples of size three are drawn without replacement from the population consisting of four numbers 3, 5, 5, 7. Find the sample mean for each sample and make a sampling distribution of the sample mean. Calculate the mean of this sampling distribution. Compare your calculations with the population mean.

2. If X_1, X_2, \dots, X_n , a random sample from $N(\mu, \sigma^2)$ and Y_1, Y_2, \dots, Y_n , a random sample from $N(0, 1)$ are independent, find the distribution of the following random variables without any derivation:

a) $\bar{X} + \bar{Y}$

b) $\frac{\sum_{i=1}^n Y_i^2}{n}$

c) $\frac{\sigma\sqrt{n} \bar{Y}}{S_X}$

d) $\frac{n\bar{Y}^2}{S_Y^2}$

e) $\sum_{i=1}^n Y_i^2 + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$

where \bar{X} and \bar{Y} are the sample means, and S_X^2 and S_Y^2 are the sample variances.

3. Let the random variable X has the density function

$$f(x) = \theta f_1(x) + (1 - \theta) f_2(x), \quad 0 < x < \infty, \quad 0 \leq \theta \leq 1$$

A single observation of the random variable X yields the value 1.

These information is also given:

- $\int_0^\infty x f_1(x) dx = 1$
- $\int_0^\infty x f_2(x) dx = 2$
- $f_2(1) = 2f_1(1) \neq 0$

Determine the method of moment estimate of θ .

4. Consider the following Pareto pdf

$$f(x; \alpha, \beta) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}, \quad x \geq \beta, \quad \alpha > 0, \quad \beta > 0$$

with mean $\frac{\alpha \beta}{\alpha - 1}$ and variance $\frac{\alpha \beta^2}{(\alpha - 1)^2 (\alpha - 2)}$.

Assume that we have a random sample of size n , X_1, X_2, \dots, X_n .

- Find the method of moment estimators of α and β .
 - Find the method of moment estimator of the 95th percentile.
5. In a military education, soldiers are divided into three teams. Their aim is to destroy a building, where terrorists used as their base, using only 3 rockets. The probability that Team A hits the building is 60%, that for Team B is 40%, and that for Team C is 80%. The team that hits the building will earn a medal. If the building is hit, which team is more **likely** to earn the medal. Considering the likelihood function, show your calculations.

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**STAT 303
MIDTERM EXAM II
QUESTIONS**

1. Let X be a random variable having pdf

$$f(x; \theta) = \frac{1}{2\theta + 1}, \quad 0 \leq x \leq 2\theta + 1, \quad \theta > 0$$

Consider a random sample, X_1, X_2, \dots, X_n .

- (8 pts.) Find the MLE of θ .
 - (7 pts.) Find the MLE of the median.
2. (10 pts.) Let X_1, X_2, \dots, X_n be a random sample from a discrete distribution with support equal to $\{0, 1, 2, 3\}$. Suppose that θ can only take on the values $\theta = 0$ and $\theta = 1$. The PMFs for $\theta = 0$ and $\theta = 1$ are:

	$\theta=0$	$\theta=1$
$x=0$	0.1	0.2
$x=1$	0.3	0.4
$x=2$	0.3	0.3
$x=3$	0.3	0.1

Suppose that $n = 6$ and the data is 0, 3, 1, 2, 0, 3. Find the MLE of θ .

3. Let X_1, X_2, \dots, X_n be a r.s. from the pdf

$$f(x; \alpha, \beta) = \frac{1}{\beta} e^{-\frac{(x-\alpha)}{\beta}}, \alpha \leq x < \infty, -\infty < \alpha < \infty, \beta > 0.$$

- (10 pts.) Find the MLE of α and β .
- (10 pts.) Find the MLE of $P(X_1 \leq 2)$.

4. Let (X_1, \dots, X_n) be a random sample from the distribution with pdf

$$f(x; a) = 3a^3 x^{-4}, \quad a \leq x,$$

where $a > 0$.

- PDF of $X_{(n)}$: $g_{X_{(n)}}(y) = nF_X(y)^{n-1} f_X(y), -\infty < y < \infty$
- PDF of $X_{(1)}$: $g_{X_{(1)}}(y) = n[1 - F_X(y)]^{n-1} f_X(y), -\infty < y < \infty$
- Leibnitz's Rule:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x; t) dx = f(b(t); t) \left(\frac{d}{dt} [b(t)] \right) - f(a(t); t) \left(\frac{d}{dt} [a(t)] \right) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x; t) dx$$

- (10 pts.) Find a sufficient statistic for a .
- (10 pts.) Show that the statistic that you found in part a) is a complete sufficient statistic for a .
- (10 pts.) Obtain the unique minimum variance unbiased estimator of a , if exists.

5. Let X_1, X_2, \dots, X_n be a r.s. of size n from a distribution with pdf

$$f(x; \theta) = \theta(1+x)^{-(1+\theta)}, x > 0$$

- (5 pts.) Find a complete sufficient statistic for θ , if exists.
- (10 pts.) Find the unique minimum variance unbiased estimator of $1/\theta$, if exists.
- (10 pts.) Find the unique minimum variance unbiased estimator of θ , if exists.

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STAT 303
FINAL EXAM

QUESTIONS

1. The independent and identically distributed rvs X_1, X_2, \dots, X_n have the common distribution

$$f(x; \alpha, \beta) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}, x \geq \beta$$

where α and β are positive.

- a) (10 pts.) Find the MLEs of α and β .
- b) (10 pts.) Find the joint minimal sufficient statistics of α and β .

2. Let X be a random variable having pdf

$$f(x; \theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta}, 0 < x < \infty, \theta > 0$$

Consider a random sample, X_1, X_2, \dots, X_n .

- a) (5 pts.) Find the Fisher Information in a random variable, $I(\theta)$.
 - b) (5 pts.) Find the Cramer Rao Lower Bound for θ .
 - c) (5 pts.) Find the efficient estimator of θ , if it exists.
 - d) (5 pts.) Find the efficient estimator of θ^2 , if it exists. If it does not exist, find the asymptotically efficient estimator of θ^2 and specify its asymptotic distribution.
3. Let X_1, X_2, \dots, X_n be a r.s. of size n from a distribution with pdf

$$f(x; \theta) = \theta(1+x)^{-(1+\theta)}, x > 0$$

- a) (5 pts.) Find an MLE of θ , if it exists.

- b) (5 pts.) Find a complete sufficient statistic for θ , if exists.
- c) (5 pts.) Find CRLB for $1/\theta$.
- d) (5 pts.) Find the MVUE of $1/\theta$.
- e) (5 pts.) Is the MVUE of $1/\theta$ also the efficient estimator for $1/\theta$.
- f) (5 pts.) Find the asymptotic distribution for the MLE of $1/\theta$.

4. Suppose that $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$ are independent random variables and we have random samples of size n , X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_n . (Do not need to derive the distributions. Just use the properties of Normal distribution).

a) (5 pts.) What is the distribution of $\bar{X} + \bar{Y}$ where \bar{X} and \bar{Y} are the sample means?

b) (5 pts.) What is the distribution of $\frac{(n-1)(S_X^2 + S_Y^2)}{\sigma^2}$ where

$S_X^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$ and $S_Y^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$ are the sample standard deviations?

c) (5 pts.) Find the expectation of $E(S_X^2 + S_Y^2)$

5. Please answer the following questions.

- a) (5 pts.) What is a random variable? Explain briefly.
- b) (5 pts.) What is the meaning of Cramer Rao Lower Bound? Why we need them?
- c) (5 pts.) Which information can be obtained from the Fisher Information? Explain the logic behind the calculations of the Fisher Information.