points of the line in the usual way. To each point is assigned a real number; the numerical difference between two coordinates is taken to be the distance between those two points, and also the length of the interval of which they are the end points. It is clear, for example, that the measure of all of the points between zero and one—the measure of the interval [0,1]—is equal to one. Since the set of all rational numbers is denumerable, the measure of all of the points in that interval with rational coordinates is zero. The measure of all of the irrational points between zero and one is, therefore, by subtraction, one. This set, of course, is not an interval, nor is it the union of any finite or denumerable set of intervals. It is doubtful that we could properly refer to the "length" of any such set; nevertheless, it does have a well-defined measure. We see that this concept of measure is, indeed, a generalization of the concept of length. However, not all sets of points on the line have measures; for reasons we need not go into, some sets are not measurable, and they receive no measure at all (which is *not* to say that their measure is zero, for zero is a very definite measure).

It is important to emphasize the fact that measure theory does not represent merely an extension of ordinary arithmetical addition (including the summation of infinite series) to the addition of non-denumerable sets of terms. In elementary arithmetic, if we are given a set of terms, say 2, 3, 5, it has the unique sum 10. Given the same set of terms again, the sum must be the same once more. In ordinary addition, even the order of the terms does not matter, but in dealing with infinite series the order of the terms may make a difference.[31] However, given the same infinite set of terms in the same order, the sum must always be the same. For example, our series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

has the unique sum 1, and the infinite series

$$0 + 0 + 0 + \ldots$$

has the unique sum 0.

Measures do not behave in the same way. As Cantor showed, any line segment of any length with its end points removed has precisely the same number of points as any other, and the infinite straight line also has the same number *c*. Moreover, the points composing any such open interval or entire line have precisely the same internal ordering amongst themselves. This can be shown by a simple diagrammatic argument (see Figure 5). Given two line segments AB and CD of unequal length, we may place the shorter above the longer and connect the end points of AB and CD with lines that intersect at point P. Using P as a point of projection, we can connect any point in

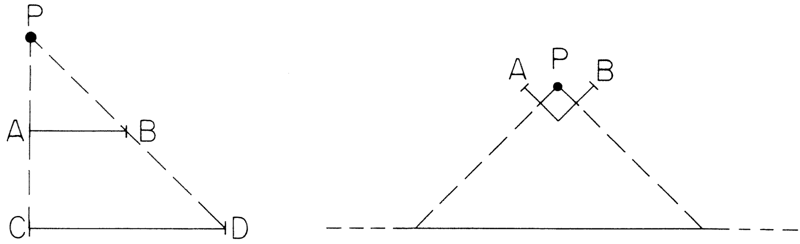

FIGURE 5.

the open interval (AB) to a point in the open interval (CD) by a line through P, and we can similarly connect any point in (CD) to a point in (AB). This shows that there must be the same number of points in (AB) and (CD), for we have just shown how to establish a one-to-one correspondence between the members of the two sets of points. By breaking the segment AB, we can show by similar reasoning that the open interval (AB) has the same number of points as the infinite line. Moreover, this correspondence between the points on the two lines is order-preserving; that is, if two points *a* and *b* in (AB) correspond respectively to two points *c* and *d* of (CD), then if *a* is to the left of *b* we will find *c* to the left of *d*. The existence of such an order-preserving one-to-one correspondence is the defining characteristic of sameness of order; two sets that have the same order in this precisely defined sense are said to be *isomorphic* to one another. Thus, we see that every open interval, finite or infinite, is isomorphic to every other.

It is an immediate consequence of these facts that the measure of an interval is not uniquely determined by the number of points it contains and the order in which they occur. Hence, if we assign each point measure zero, and attempt to "sum" them in the order in which they occur, we find that a given set of terms in a given order does *not* determine a unique "sum." The measure of a set of points depends upon more than the size (measure) of each of the points and the order in which they occur.

We have just seen that point sets containing *c* elements could have any finite length (measure) greater than zero, or infinite length. We have also seen that any set of points with a finite or denumerably infinite number of members must have zero length (measure). To prevent the tempting misconception that the measure of a set of points is greater than zero if and only if it has cardinality *c*, let us consider Cantor's ingenious *discontinuum*; it contains *c* points, but has measure zero. We begin with a line segment, say the set of points between zero and one, end points included. We remove the middle