4. (15 Points) If G is a group and X a finite symmetric generating set of G, the *growth function* of G with respect to X is the function $f_{(G,X)} : \mathbb{N} \to \mathbb{N}$ defined by

$$f_{(G,X)}(n) = |\{g \in G \mid |g|_X \leq n\}|$$

Let G be a group and X a finite symmetric generating set of G. Show that G is infinite if and only if $f_{(G,X)}$ is monotone increasing.

($\Leftarrow$) If $f_{(G,X)}$ is monotone increasing,

$\forall n \geq 1 \ \exists g_n \in G$ s.t $|g_n|_X = n$.

So, $\{g_i \mid i \geq 1\}$ is an infinite subset of G.

($\Rightarrow$) let $B(n) = \{g \in G \mid |g|_X \leq n\}$ so that $f_{(G,X)}(n) = |B(n)|$.

$B(0) \subseteq B(1) \subseteq \dots B(n) \subseteq B(n+1) \subseteq \dots$ with $G = \bigcup_{n=0}^{\infty} B(n)$.

if $f_{(G,X)}$ is not monotone increasing, $\exists N$ s.t.

$f_{(G,X)}(N) = f_{G(X)}(N+i) \quad \forall i = 1, 2, \dots$

$\Rightarrow B(N) = B(N+i) \quad \forall i = 1, 2, \dots$

$\Rightarrow G = \bigcup_{n=0}^{N} B(n) \Rightarrow G$ is finite.

5. (15 Points) Show that $\mathbb{Z} \times D_\infty \cong \langle a, b, c, \mid c^2, (cb)^2, aba^{-1}b^{-1}, aca^{-1}c^{-1} \rangle = G$

let $f: F(a,b,c) \to \mathbb{Z} \times D_\infty$

$a \mapsto (1, e)$

$b \mapsto (0, r)$

$c \mapsto (0, s)$

, $f(c^2) = (0, s^2) = (0, e)$

$f((cb)^2) = (0, (sr)^2) = (0, e)$

$f(aba^{-1}b^{-1}) = (0, e)$

$f(aca^{-1}c^{-1}) = (0, e)$

Note in G:

$ab = ba$

$ac = ca$

$cb = b^{-1}c$

So, $\exists$ gp hom $\bar{f}: G \to \mathbb{Z} \times D_\infty$

$a \mapsto (1, e)$

$b \mapsto (0, r)$

$c \mapsto (0, s)$

Define $g: \mathbb{Z} \times D_\infty \to G$

$(n, r^i s^j) \mapsto a^n b^i c^j$

Claim: g is a hom.

let $x = (n, r^i s^j)$, $y = (m, r^k s^t)$

if $j = 0$

$g(xy) = g(n+m, r^{i+k} s^t)$

$= a^{n+m} b^{i+k} c^t = a^n b^i a^m b^k c^t$

$= g(x) g(y)$

if $j = 1$ $\quad g(xy) = g(n+m, r^{i-k} s^{1+t})$

$= a^{n+m} b^{i-k} c^{j+1} = a^n b^i c\, a^m b^k c^t$

$= g(x) g(y)$

Clear $\bar{f}$ and g are inverses of each other.

So $\bar{f}$ is an isom.

M E T U Department of Mathematics

Math 466 - Spring 2025 - Finaş - June 12 - 17:00 - Instructor: Gökhan Benli		
F U L L N A M E (write in CAPITAL letters)	S T U D E N T I D	S I G N A T U R E
56 QUESTIONS ON 4 PAGES	120 MINUTES	TOTAL 100 POINTS

1. (20 Points) Let G be a group and $X \subseteq G$.

(a) Let $\langle X\rangle = \bigcap_{X\subseteq H\leq G} H$. Show that $\langle X\rangle = \underbrace{\{g \in G \mid g = x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k},\ x_i \in X, n_i \in \mathbb{Z}, k \in \mathbb{N}\}}_{=K}$

($\supseteq$) Claim $K \leqslant G$.

Clearly $e \in K$. if $a = x_1^{n_1}\cdots x_k^{n_k}$, $b = y_1^{m_1}\cdots y_t^{m_t} \in K$ ($n_i \in \mathbb{Z}$, $m_j \in \mathbb{Z}$, $k, t \in \mathbb{N}$, $x_i, y_j \in X$)

$ab^{-1} = x_1^{n_1}\cdots x_k^{n_k} y_t^{-m_t}\cdots y_2^{-m_2} y_1^{-m_1} \in K$.

Also, clearly $X \subseteq K$. So, $\langle X\rangle \subseteq K$.

($\subseteq$) let $H \leqslant G$ with $X \subseteq H$. Then, since H is a subgp, $K \subseteq H$

Thus $K \subseteq \bigcap_{X\subseteq H\leqslant G} H = \langle X\rangle$

(b) Let $\langle\langle X\rangle\rangle = \bigcap_{X\subseteq H\trianglelefteq G} H$. Show that $\langle\langle X\rangle\rangle = \underbrace{\langle X^G\rangle}_{=K}$ where $X^G = \{g^{-1}xg \mid x \in X, g \in G\}$.

Clearly $X \subseteq X^G \subseteq K$.

Claim: $K \trianglelefteq G$. let $k \in K$, $g \in G$.

By part (a), $k = (g_1^{-1}x_1g_1)^{n_1}\cdots(g_t^{-1}x_tg_t)^{n_t}$ for some $n_i \in \mathbb{Z}$, $g_i \in G$, $x_i \in X$.

$= (g_1^{-1}x_1^{n_1}g_1)\cdots(g_t^{-1}x_t^{n_t}g_t)$

Then $g^{-1}kg = ((g_1g)^{-1}x_1(g_1g))^{n_1}\cdots((g_tg)^{-1}x_t(g_tg))^{n_t} \in K$. So $K \trianglelefteq G$.

Thus $\langle\langle X\rangle\rangle \subseteq K$.

let $H \trianglelefteq G$ with $X \subseteq H$. Then, since H is normal, $X^G \subseteq H$.

and hence $K = \langle X^G\rangle \subseteq H$.

Thus $K \subseteq \langle\langle X\rangle\rangle$.

2. (30 pts) (a) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\vec{u} = (1,1)$. Describe the isometry $f = T_{\vec{u}} \cdot A$.

$A = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix}$ so A is reflection wrt $y = x$.

let $\ell: y = x$

$\vec{u} \not\perp \ell \Rightarrow f$ is a glide reflection.

$(I - A)\vec{u} = 0$. So, f first reflects wrt $y = x$ and then translates by $\vec{u}$.

(b) Describe the isometry given by the matrix $B = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ -1/3 & -2/3 & 2/3 \end{bmatrix} \in SO(3)$.

$tr(B) = 2 = 2\cos\alpha + 1 \Rightarrow B$ is rotation by $\alpha = \frac{\pi}{3}$

$I - B = \begin{bmatrix} 1/3 & -1/3 & -2/3 \\ 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \end{bmatrix} \xrightarrow{\times 3} \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So, $Null(I - B) = \langle (1,-1,1) \rangle = \ell$

So, B is rotation about the line ℓ by $\frac{\pi}{3}$.

3. (20 Points) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be an orthonormal basisof $\mathbb{R}^3$ and let A be the matrix formed by taking $\vec{v}_1$ as first column, $\vec{v}_2$ as second column, and $\vec{v}_3$ as third column. Also let

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Show that ABA^{-1} represents reflection with respect to the plane containing $\vec{v}_1$ and $\vec{v}_2$, and that $-ABA^{-1}$ represents rotation by π about the axis determined by $\vec{v}_3$.

Note that A is an orthogonal matrix so $A^{-1} = A^T$. $\det(ABA^{-1}) = -1$

$A^{-1}\vec{v_1} = A^T v_1 = \begin{bmatrix} v_1 \cdot v_1 \\ v_2 \cdot v_1 \\ v_3 \cdot v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A^{-1}\vec{v_2} = A^T\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A^{-1}\vec{v_3} = A^T\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

so $(ABA^{-1})(\vec{v_1}) = AB\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v_1}$ $\quad (ABA^{-1})(\vec{v_3}) = AB\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = -\vec{v_3}$

$(ABA^{-1})(v_2) = \vec{v_2}$

So ABA^{-1} is a reflection wrt $\langle \vec{v_1}, \vec{v_2} \rangle$.

let $C = -ABA^{-1}$, $\det(C) = 1$.

$C(\vec{v_3}) = \vec{v_3}$ so C is rotation about $\langle \vec{v_3} \rangle$.

$tr(C) = -tr(ABA^{-1}) = -tr(B)$ $\Rightarrow$ C is rot by π wrt $\langle \vec{v_3} \rangle$.

$= -1 = 2\cos\alpha + 1$

(b) Find the matrix which represents reflection with respect to the plane $x + \sqrt{3}y = z$.

let $\vec{v_1} = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right)$ $\quad \{\vec{v_1}, \vec{v_2}\}$ is an orthonormal basis for the plane.

$\vec{v_2} = \left(\frac{1}{2\sqrt{5}}, \frac{\sqrt{3}}{2\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$\vec{v_3} = \left(\frac{1}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ $\quad \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is an orthonormal basis for $\mathbb{R}^3$.

(Pick any unit vector $\vec{v_1}$ on the plane and then find a unit vector $\vec{v_2}$ on the plane $\perp$ to $\vec{v_1}$. Then pick a unit vector $\vec{v_3}$ in $\langle \vec{v_1}, \vec{v_2} \rangle^{\perp}$).

By (a), if $A = \begin{bmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{5}} & 1/\sqrt{5} \\ -1/2 & \sqrt{3}/2\sqrt{5} & \sqrt{3}/\sqrt{5} \\ 0 & \sqrt{3}/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$ then

ABA^{-1} is the reflection wrt $\langle \vec{v_1}, \vec{v_2} \rangle$.