

Math 466 Exercises for Week 14

June 10, 2025

1. Let G be a group and $X \subseteq G$. Let $\langle X \rangle$ be the intersection of all subgroups of G which contain X , that is

$$\langle X \rangle = \bigcap_{X \subseteq H \leq G} H$$

($\langle X \rangle$ is the subgroup generated by X). Show that $\langle X \rangle = \left\{ g \in G \mid g = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}, x_i \in X, n_i \in \mathbb{Z}, k \in \mathbb{N} \right\}$

2. Let G be a group and $X \subseteq G$. Let $\langle\langle X \rangle\rangle$ be the intersection of all **normal** subgroups of G which contain X , that is

$$\langle\langle X \rangle\rangle = \bigcap_{X \subseteq H \trianglelefteq G} H$$

($\langle\langle X \rangle\rangle$ is the normal subgroup generated by X). Let X^G denote the set of all conjugates of elements of X , that is

$$X^G = \{g^{-1}xg \mid x \in X, g \in G\}$$

Show that $\langle\langle X \rangle\rangle = \langle X^G \rangle$

3. Let F_2 be the free group on two generators. Show that $F_2 \times F_2$ is not a free group.
4. Show that $D_\infty \cong \langle r, s \mid s^2, (sr)^2 \rangle$
5. Show that $D_\infty \cong \langle a, b \mid a^2, b^2 \rangle$ (Hint: Use ex 4)

If G is a group and X a finite symmetric generating set of G , the *growth function* of G with respect to X is the function $f_{(G,X)} : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$f_{(G,X)}(n) = |\{g \in G \mid |g|_X \leq n\}|$$

6. Consider \mathbb{Z} with generating sets $X = \{\mp 1\}$ and $Y = \{\mp 2, \mp 3\}$. Find the functions $f_{(\mathbb{Z},X)}$ and $f_{(\mathbb{Z},Y)}$.
7. Consider D_∞ with generating set $X = \{r, r^{-1}, s\}$. Find $f_{(D_\infty, X)}$.
8. Let $F_2 = F(a, b)$ be the free group on two generators and $X = \{\mp a, \mp b\}$. Find $f_{(F_2, X)}$.
9. Let G be a group and X a finite symmetric generating set of G . Show that G is infinite if and only if $f_{(G,X)}$ is monotone increasing.