Free gps. and group presentations:

Consider D_n, $n \geq 2$

it is generated by r and s subject to "relations"

$$r^n = e, \quad s^2 = e, \quad sr = r^{-1}s \ (\Leftrightarrow (sr)^2 = e)$$

Now let G be a group generated by a, b

with $\underbrace{a^n = e, \ b^2 = e}_{(*)}$, $\underbrace{(ba)^2 = e}_{(**)}$ $(\Leftrightarrow ba = a^{-1}b)$

an arbitrary element of G is of the form:

$$g = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} \qquad m_i, n_i \in \mathbb{Z}.$$

by using $(*)$ we may assume $m_i \in \{0, .., n-1\}$, $n_i \in \{0, 1\}$

also, by using $(**)$ we can write

$$g = a^t b^e \qquad t \in \{0, .., n-1\}, \ e \in \{0, 1\}$$

φ is surjective

Define: $\varphi : D_n \longrightarrow G$

$$r^t s^e \longmapsto a^t b^e$$

Also, $(**) \Rightarrow \varphi$ is a gp. hom.

So, $D_n / \ker\varphi \cong G$.

So, any group G gen. by a and b satisfying $(*)$ and $(**)$

is a quotient of D_n.

We indicate this by: $D_n \cong \langle \underbrace{r, s}_{\text{generators}} \mid \underbrace{r^n, s^2, (sr)^2}_{\text{relations}} \rangle$

we will make this precise.

Free Groups:

"free of relations"

intuitively: A group G is generated "freely" by $x_1, \dots, x_n$ if there is "no non-trivial relations" among $x_1, \dots, x_n$

for ex: D_n is NOT freely generated by r and s since $sr = r^{-1}s$ is a non-trivial relation.

or $\mathbb{Z}_n$ is not freely gen. by $\bar{1}$ since $n \cdot \bar{1} = \bar{0}$.

But $\mathbb{Z}$ is freely generated by 1.

Let X be non-empty set.

A "word" on X is a finite sequence

$$x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}, \quad x_i \in X, \; m_i \in \mathbb{Z}.$$

also the "empty word" e is a word.

let X^* be the set of words over X.

the word is "reduced" if $x_i \neq x_{i+1}$ $i = 1, \dots, k-1$

and $m_i \neq 0$ $i = 1, \dots, k$

OR the word is the empty word e.

Rk: Every word over X can be "simplified" to a reduced word.

ex: $X = \{x, y, z\}$

$w = z^0 x^{-3} x^{-2} y^5 x^7 z^2 z^{-2} x^{-1} x y y^2 x^{-1}$ is word over X
not reduced

↓

reduced version

$\overline{w} = x^{-5} y^5 x^7 y^3 x^{-1}$

Prop: Each word over X can be simplified to a unique reduced word.

Pf: Ex.

let $F(X)$ denote the set of reduced words over X.

if $u, v \in F(X)$ define $u \cdot v = \overline{uv}$, this gives a bin op on $F(X)$.

↓ concatenation and then simplification.

Claim: $F(X)$ is a gp. with this operation

assoc. $\overline{u\overline{vw}}$ and $\overline{\overline{uv}w}$ are both reductions of uvw so are equal.

id: e

inverses $(x_1^{m_1} \cdots x_k^{m_k})^{-1} = x_k^{-m_k} \cdots x_2^{-m_2} x_1^{-m_1}$

$F(X)$ is the "free gp. on X".

<u>ex</u>: $X = \{x, y, z\}$

$u = x^{-1} y^{2} z^{3} \in F(X)$

$v = z^{-3} y^{-1} x^{-10} z$

$u \cdot v = x^{-1} y^{2} \cancel{z^{3}} \cancel{z^{-3}} y^{-1} x^{-10} z = x^{-1} y^{1} x^{-10} z$

We will mostly have $|X| < \infty$.

<u>ex</u>: $X = \{x\}$ ie $|X| = 1$

reduced words are x^m, $m \in \mathbb{Z}$ $(x^0 = e)$

Clearly $F(X) \cong \mathbb{Z}$

<u>ex</u>: if $X = \{a, b, \dots\}$ ie $|X| \geq 2$

in $F(X)$ $ab \neq ba$ ie $F(X)$ is non-abelian.

<u>Rk</u>: 1) if $|X| = |Y|$ then $F(X) \cong F(Y)$

2) Fact: $F(X) \cong F(Y) \iff |X| = |Y|$.

So, if $|X| = n$ let us write F_n for $F(X)$

"free gp with n generators"

ex: There is no non-trivial hom. from $\mathbb{Z}_n$ to $\mathbb{Z}$.

Pf: Suppose $f: \mathbb{Z}_n \to \mathbb{Z}$ is non-trivial,

then $f(\bar{1}) = k \neq 0$

$\Rightarrow$ $f(n \cdot \bar{1}) = n \cdot k \neq 0$ a contradiction.

$\quad \parallel$

$f(\bar{0}) = 0$

Thm: let $F(X)$ be the free gp. on X.

Given any group G and and function $f: X \to G$ then f extends uniquely to a gp. hom. $\bar{f}: F(X) \to G$ s.t $\bar{f}(x) = f(x)$ $\forall x \in X$.

(Compare this with vector spaces, bases and linear trans)

Pf: given $u = x_1^{m_1} \cdots x_k^{m_k} \in F(X)$

define $\bar{f}(u) = f(x_1)^{m_1} \cdots f(x_k)^{m_k}$, $\bar{f}(e) = e_G$

clearly $\bar{f}$ extend f and is a gp. hom. $\blacksquare$

Corollary: If G is generated by $g_1, \dots, g_n$ then $\exists$ a surjective hom $\varphi: F_n \longrightarrow G$, $x_i \longmapsto g_i$

i.e. $F_n / \ker\varphi \cong G$.

i.e "F_n is the largest group which can be generated by n elements".

Defn: A group G is "freely generated by X" if every $g \in G - \{e\}$ can be written uniquely as a reduced word $g = x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$ $\quad x_i \neq x_{i+1}$ $i = 1, \dots, k-1$, $m_i \neq 0$

Cor: A group G is generated freely by X $\iff G \cong F(X)$

Pf: $(\Leftarrow)$ Clear

$(\Rightarrow)$ $f: X \to G$, $x \longmapsto x$ extends to a gp. hom $\bar{f}: F(X) \longrightarrow G$

Since X generates G, $\bar{f}$ is surjective

since X generates G freely, $\bar{f}$ is injective. $\blacksquare$

ex: $G = S_n$, $X =$ transpositions

X does not generate S_n freely since $(12) = (12)(12)(12)$

$x = (12)$

$x = x^3$

a reduced word over X

$G = \mathbb{Z} \oplus \mathbb{Z}$, $X = \{ \overset{x}{(1,0)}, \overset{y}{(0,1)} \}$

does not generate G freely since

$$(1,0) + (0,1) = (0,1) + (1,0)$$

$$xy = yx$$

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ex: $G = D_n$, $X = \{r, s\}$ $\quad F_2 = F(X)$

let $\varphi : F_2 \longrightarrow D_n$ be the surjective hom. of the thm.

$$r \longmapsto r$$
$$s \longmapsto s$$

So, $F_2 / \ker\varphi \cong D_n$ $\quad$ let $\ker\varphi = N \trianglelefteq F_2$.

Q: Can we describe N?

Relations of D_n: r^n, s^2, $(sr)^2$

let M be the smallest normal subgroup of F_2 containing $r^n, s^2, (sr)^2$

i.e. $$M = \bigcap_{\substack{K \trianglelefteq F_2 \\ r^n, s^2, (sr)^2 \in K}} K \trianglelefteq F_2$$

Claim: $M = N$

$(\subseteq)$ Since $\varphi(r^n) = r^n = e$

$\varphi(s^2) = s^2 = e$ $\Rightarrow r^n, s^2, (sr)^2 \in N = \ker\varphi.$

$\varphi((sr)^2) = (sr)^2 = e$

So, $M \subseteq N.$

$(\supseteq)$ Since $M \subseteq N$

the function $F_2/M \longrightarrow F_2/N$ is well ~~de~~ defined

$rM \longrightarrow rN$

and surjective so $\left|F_2/M\right| \geq \left|F_2/N\right| = |D_n| = 2n$

Since $r^n, s^2, (sr)^2 \in M$, any element of F_2/M

can be written as $(rM)^i (sM)^j$ $i \in \{0, \dots, n-1\}$

$j \in \{0, 1\}$

[ex: $(r^{n+3}M)(s^3M)(rM) = (r^3M)(sM)(rM)$]

$= (r^3M)(srM)$

$= (r^3M)(r^{-1}sM)$

$= (r^2M)(sM)$

So $\left|F_2/M\right| \leq 2n$

Hence $\left|{F_2}/{M}\right| = 2n \implies M = N$.

Conclusion: $D_n \cong {F_2}/{M}$ $\quad F_2 = F(r,s)$

$M =$ smallest normal subgp of F_2 containg $r^n, s^2, (rs)^2$

We write this as:

$$D_n \cong \langle \underbrace{r, s}_{\text{generators}} \mid \underbrace{r^n, s^2, (rs)^2}_{\text{relations}} \rangle$$

this is a "presentation" of D_n.

Defn: let X be a non-empty set, $R \subseteq F(X)$

The "group defined by gerators X and relations R" is $G = {F(X)}/{M}$ where M is the smallest normal subgroup of $F(X)$ containg R.

We write this as $G \cong \langle X \mid R \rangle \to$ a "group presentation"

if $X = \{x_1, \dots, x_n\}$ $\quad R = \{r_1, \dots, r_m\} \subseteq F(x_1, \dots, x_n)$

$$G \cong \langle x_1, \dots, x_n \mid r_1, r_2, \dots, r_m \rangle = \langle x_1, \dots, x_n \mid r_1 = r_2 = \dots = r_m = e \rangle$$

So, above we have proven:

$$D_n \cong \langle r, s \mid r^n, s^2, (rs)^2 \rangle$$ → a presentation for D_n.

$$= \langle r, s \mid r^n = e, s^2 = e, (rs)^2 = e \rangle$$