4. (15 Points) Suppose a group G acts on a finite set X. Show that if $g, h \in G$ are conjugate, then g and h fix the same number of elements of X, i.e, $|Fix(g)| = |Fix(h)|$.

Suppose $g = k^{-1} h k$ for some $k \in G$.

$x \in Fix(g) \Leftrightarrow g.x = x \Leftrightarrow k^{-1} h k . x = x \Leftrightarrow h(k.x) = kx \Leftrightarrow kx \in Fix(h)$

So, define $\varphi : Fix(g) \longrightarrow Fix(h)$

$x \longmapsto k.x$

$k.x = k.y \Rightarrow x = y \Rightarrow \varphi$ is inj.

if $y \in Fix(h)$, $k^{-1} y \in Fix(g)$ and $\varphi(k^{-1} y) = y \Rightarrow \varphi$ is surj.

5. (15 Points) Using the classification of finite subgroups of $SO(3)$, show that A_5 has no subgroups of order 30. (Hint: A_5 has no element of order 15).

$\left.\begin{array}{l} \mathbb{Z}_n, \ n \geq 1 \\ D_n \ \ n \geq 2 \\ A_4 \\ S_4 \\ A_5 \end{array}\right\}$ possibilities of finite subgps of $SO(3)$

let $H \leq A_5$ with $|H| = 30$

so $H \leq SO(3)$ of order 30.

$\Rightarrow H \cong \mathbb{Z}_{30}$ OR $H = D_{15}$

But Both have elements of order 15.

A_5 does not have elements of order 15:

$\left(A_5 = e, \underbrace{20 \text{ 3-cycles}}_{\text{order } 3}, \underbrace{24 - 5\text{cycles}}_{\text{order } 5}, \underbrace{15 \text{ elts of the form } (ab)(cd)}_{\text{order } 2} \right)$

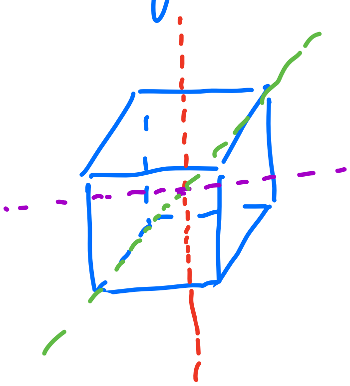
M E T U Department of Mathematics

Math 466 - Spring 2025 - Midterm 2 - May 16 - 17:40 - Instructor: Gökhan Benli		
F U L L N A M E (**write in CAPITAL letters**)	S T U D E N T I D	S I G N A T U R E
6 QUESTIONS ON 4 PAGES	120 MINUTES	TOTAL 100 POINTS

1. (20 Points) Let $C \subseteq \mathbb{R}^3$ be the cube whose vertices are at $(\mp 1, \mp 1, \mp 1)$ and let $G \leq SO(3)$ be the group of rotations of C. Also, let $L = \{\ell_1, \ell_2, \ell_3\}$ be the x, y, z axes respectively. Clearly G acts on L. Hence we have a homomorphism $\varphi : G \to S_3$.

(a) Show that φ is surjective.

Pick two opposite faces and let ℓ be the line passing through midpoints of faces. (ℓ is one of the x, y, z axis.)



Let r be the rotation about ℓ by $\pi/2$ then $r(\ell) = \ell$ and r swaps the two other axes. This shows any transposition is in the image of φ. Since transpositions generate S_3, $\varphi(G) = S_3$.

(b) Show that the kernel of φ has 4 elements and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$|G| = 24$ and $|\varphi(G)| = 6$ so $|\mathrm{Ker}\,\varphi| = 4$.

Let r_x, r_y, r_z be rotations by π about each axes.

Clearly, $r_x, r_y, r_z \in \mathrm{Ker}(\varphi)$ and $r_x^2 = r_y^2 = r_z^2 = e$.

So $\mathrm{Ker}\,\varphi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

2. (20 pts) Let $A, B \in SO(3)$ be two rotation matrices with axes ℓ_A, ℓ_B and angles α_A, α_B respectively. Show that $AB = BA$ if and only if $(\ell_A = \ell_B)$ or $(\ell_A \perp \ell_B$ and $\alpha_A = \alpha_B = \pi.)$

($\Rightarrow$) Suppose $AB = BA$.

let $A\vec{v} = \vec{v}$, $\vec{v} \neq \vec{0}$

B preserves lengths

Then $A(B\vec{v}) = B(A\vec{v}) = B\vec{v} \Rightarrow B\vec{v} \parallel \vec{v} \Rightarrow B\vec{v} = \mp\vec{v}$

if $B\vec{v} = \vec{v} \Rightarrow \ell_A = \ell_B$

if $B\vec{v} = -\vec{v} \Rightarrow \underbrace{\ell_B = \langle \vec{v} \rangle^{\perp}}_{\Leftrightarrow\, \ell_A \perp \ell_B}$ and $\alpha_B = \pi$

($\Leftarrow$) Clear if $\ell_A = \ell_B$.

if $\ell_A \perp \ell_B$ and $\alpha_A = \alpha_B = \pi$

let $\ell_A = \langle \vec{u} \rangle$, $\ell_B = \langle \vec{v} \rangle$ so that $\vec{u} \perp \vec{v}$. let $\vec{w} \in \langle \vec{u}, \vec{v} \rangle^{\perp}$, $\vec{w} \neq \vec{0}$

$$\left.\begin{aligned}
(AB)(\vec{u}) &= A(-\vec{u}) = -A\vec{u} = -\vec{u} \\
(BA)(\vec{u}) &= B(\vec{u}) = -\vec{u} \\
(AB)(\vec{v}) &= A(\vec{v}) = -\vec{v} \\
(BA)(\vec{v}) &= B(-\vec{v}) = -\vec{v} \\
(AB)(\vec{w}) &= A(-\vec{w}) = \vec{w} \\
(BA)(\vec{w}) &= B(-\vec{w}) = \vec{w}
\end{aligned}\right\}$$

AB and BA agree on the basis $\{\vec{u}, \vec{v}, \vec{w}\}$

$\Rightarrow AB = BA$

3. (15+15=30 Points) (a) Let $A \in SO(3)$ be a rotation matrix. Show that the rotation angle α of A satisfies $1 + 2\cos(\alpha) = tr(A)$. (Here, $tr(A)$ is the trace of A, you may use the fact that conjugate matrices have the same trace without a proof.)

let $A\vec{u} = \vec{u}$ $\|\vec{u}\| = 1$ and let $\{\vec{v}, \vec{w}\}$ be an orthonormal basis of $\langle \vec{u} \rangle^{\perp}$.

Then $B = \{\vec{v}, \vec{w}, \vec{u}\}$ is an o.n. basis of $\mathbb{R}^3$.

$A\vec{v} \in \langle \vec{v}, \vec{w} \rangle$
$A\vec{w} \in \langle \vec{v}, \vec{w} \rangle$
$A\vec{u} = \vec{u}$

$$P^{-1}AP = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ for some $a, b, c, d \in \mathbb{R}$. where $P = \begin{bmatrix} \vec{v} & \vec{w} & \vec{u} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

$1 = \det A = \det P^{-1}AP = ad - bc \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2)$

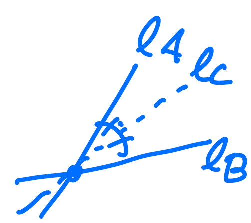
$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ where $\alpha = \alpha_A$.

So, $tr(A) = tr(P^{-1}AP) = 2\cos\alpha + 1$

(b) Let $A, B \in SO(3)$. Show that A and B are conjugate if and only if $tr(A) = tr(B)$.

$(\Rightarrow)$ ✓

$(\Leftarrow)$ suppose $tr(A) = tr(B) \Rightarrow \alpha_A = \alpha_B$



let ℓ_A, ℓ_B be the axes of rotations.

let $C \in SO(3)$ be a rotation s.t $C(\ell_A) = \ell_B$

(C is a rotation about ℓ_C)

<u>Claim:</u> $CAC^{-1} = B$

if $\ell_B = \langle \vec{u} \rangle$, $(CAC^{-1})(\vec{u}) = CA(\underbrace{C^{-1}\vec{u}}_{\in \ell_A}) = C(C^{-1}\vec{u}) = \vec{u}$

So B and CAC^{-1} have same axis of rotation.

Also $\alpha_B = \alpha_A = \alpha_{CAC^{-1}}$ hence B and CAC^{-1} have same angle of rot. $\Rightarrow B = CAC^{-1}$.