Math 466 Exercises for Week 11

May 12, 2025

1. Using arguments done in class, show that the symmetry group of the dodecahedron has 120 elements. (The exact structure will be shown in class next week).

2. Suppose we want to make a dice out of a cube by writing the numbers from 1 to 6 on each side. Using Burnside's theorem (as in exercise 4 of week 10), find all possible such dice. Do this also for the other platonic solids. (You need to use the facts that the tetrahedron has 12 rotation symmetries, the cube has 24 rotation symmetries and the dodecahedron has 60 rotation symmetries, this last fact will be proven in class.)

3. Show that S_n is generated by the permutations $(1, 2, 3, \ldots, n)$ and $(1, 2)$.

4. Let $L = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be the "principal axes" of the cube, i.e., lines which pass through opposite vertices of the cube. Also let G^+ be the group of rotational symmetries of the cube.

   (a) Show that any rotational symmetry symmetry of the cube permutes elements of L and hence we have an action of G^+ on L.

   (b) Show that this action is faithful , that is if $g \in G^+$ satisfies $g(\ell_i) = \ell_i$ for all $i = 1, 2, 3, 4$, then $g = id$. Hence we have an injective homomorphism $\phi : G^+ \to S_4$.

   (c) Show that $(1, 2)$ and $(1, 2, 3, 4)$ are in the image of ϕ and hence (by ex 3), ϕ is an isomorphism.

5. Let G be the isometry group of the cube.

   (a) Show that inscribed into the cube, there are 2 tetrahedra T_1, T_2, whose vertices are vertices of the cube. (Number the vertices of the cube and describe the vertices of T_1, T_2 by this numbering)

   (b) Show that if f is an isometry of the cube, then $f(T_1) = T_1, f(T_2) = T_2$ or $f(T_1) = T_2, f(T_2) = T_1$.

   (c) Let $J : \mathbb{R}^3 \to \mathbb{R}^3$ given by $J(x) = -x$ and note that J is an isometry of the cube. Show that J interchanges T_1 and T_2.

   (d) Let H be the isometry group of the tetrahedron (Recall $H \cong S_4$). Define $\phi : G \to H \times \{\mp 1\}$ as follows:

   $$\phi(f) = \left\{ \begin{array}{ccc} (f, 1) & \text{if} & f(T_1) = T_1 \\ (fJ, -11) & \text{if} & f(T_1) = T_2 \end{array} \right\}$$

   and show that ϕ is an isomorphism.