Math 466 Exercises for Week 10

May 4, 2025

1. Suppose a group G acts on a set X. If $g \cdot x = y$, show that $gStab_G(x)g^{-1} = Stab_G(y)$.

2. Let G be a group and consider the action of G onto itself given by $g \cdot h = gh$ for every $g, h \in G$. Show that the homomorphism $\phi : G \to Sym(G)$ $g \mapsto f_g$ where $f_g(h) = g \cdot h \ \forall \ h \in G$, is injective. Conclude that if $|G| = n$, then G is isomorphic to a subgroup of S_n.

3. (a) a) Using the Orbit-Stabilizer Theorem, show that the symmetry group of a regular n-gon in $\mathbb{R}^2$ has $2n$ elements. (Think of the action on the sides of the n-gon.)

   (b) Using the Orbit-Stabilizer Theorem, show that the symmetry group of a cube in $\mathbb{R}^3$ has 48 elements. (Think of the action on the faces of the cube).

4. Suppose we want count the number of different colorings of the sides of a regular n-gon with n different colors (up to rotation). For example the following colorings of the regular pentagon are the same:
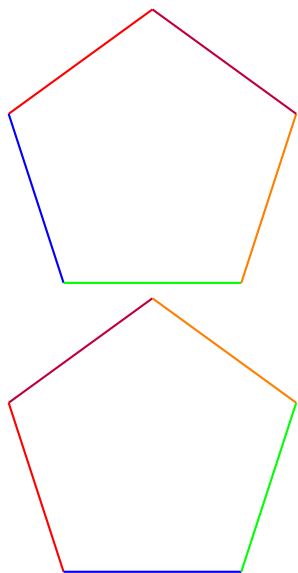
Think of $\mathbb{Z}_n$ acting on the set of all colorings by rotations. Using Burnside's Theorem about the number of orbits of an action, find the number of all possible colorings of a regular n-gon up to rotation.